

# CRITICAL BEHAVIOURS OF THE FIFTH PAINLEVÉ TRANSCENDENTS AND THE MONODROMY DATA

SHUN SHIMOMURA

**ABSTRACT.** For the fifth Painlevé equation, we present families of convergent series solutions near the origin and the corresponding monodromy data for the associated isomonodromy linear system. These solutions are of complex power type, of inverse logarithmic type and of Taylor series type. It is also possible to compute the monodromy data in non-generic cases. Solutions of logarithmic type are derived from those of inverse logarithmic type through a Bäcklund transformation found by Gromak. In a special case the complex power type of solutions have relatively simple oscillatory expressions. For the complex power type of solutions in the generic case, we clarify the structure of the analytic continuation on the universal covering around the origin, and examine the distribution of zeros, poles and 1-points. It is shown that two kinds of spiral domains including a sector as a special case are alternately arrayed; the domains of one kind contain sequences both of zeros and of poles, and those of the other kind sequences of 1-points.

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## 1. INTRODUCTION

For the solutions of the sixth Painlevé equation Guzzetti [9] provided the tables of their critical behaviours and parametric connection formulas. The solutions near each critical point are classified as follows: complex power type, logarithmic type, inverse oscillatory type, inverse logarithmic type, and Taylor series type. All of them are derived through birational transformations from four basic solutions, two of which are of complex power type and two are of logarithmic type. These tables consist of important formulas as nonlinear special functions and are expected to be of great use in applications to a variety of problems in mathematics and mathematical physics.

The fifth Painlevé equation normalised in the form

$$(V) \quad \frac{d^2 y}{dx^2} = \left( \frac{1}{2y} + \frac{1}{y-1} \right) \left( \frac{dy}{dx} \right)^2 - \frac{1}{x} \frac{dy}{dx} \\ + \frac{(y-1)^2}{8x^2} \left( (\theta_0 - \theta_x + \theta_\infty)^2 y - \frac{(\theta_0 - \theta_x - \theta_\infty)^2}{y} \right) + (1 - \theta_0 - \theta_x) \frac{y}{x} - \frac{y(y+1)}{2(y-1)}$$

with  $\theta_0, \theta_x, \theta_\infty \in \mathbb{C}$  follows from the isomonodromy deformation of a 2-dimensional linear system of the form

$$(1.1) \quad \frac{dY}{d\lambda} = \left( \frac{A_0(x)}{\lambda} + \frac{A_x(x)}{\lambda - x} + \frac{J}{2} \right) Y,$$

$J = \text{diag}[1, -1]$ , where  $A_0(x)$  and  $A_x(x)$  satisfy the following:

- (a) the eigenvalues of  $A_0(x)$  and  $A_x(x)$  are  $\pm\theta_0/2$  and  $\pm\theta_x/2$ , respectively;
- (b)  $(A_0(x) + A_x(x))_{11} = -(A_0(x) + A_x(x))_{22} \equiv -\theta_\infty/2$

(cf. [12, §3], [13, Appendix C]). Jimbo [12] studied the asymptotic behaviour of the  $\tau$ -function for (V) near  $x = 0$  parametrised by its corresponding monodromy data for (1.1). Applying WKB analysis to (1.1), Andreev and Kitaev [2] obtained asymptotic solutions of (V) along the real axis near  $x = 0$  and  $x = \infty$  together with their monodromy data that yield the connection formulas between them. For more general integration constants, a family of solutions near  $x = 0$  expanded into convergent series in spiral domains or sectors was given by the present author [18]. Kaneko and Ohyama [14] presented certain Taylor series solutions around  $x = 0$  such that each corresponding linear system (1.1) is solvable in terms of hypergeometric functions and that the monodromy may be explicitly calculated.

For the fifth Painlevé transcendents as well, it is preferable to give tables of critical behaviours like those of Guzzetti [9]. Toward this goal, in this paper, near the critical point  $x = 0$  of (V), we present families of solutions expanded into convergent series of three types and the respective monodromy data parametrised by integration constants yielding analogues of the parametric connection formulas in [9]. The monodromy data are given under more general conditions on  $\theta_0, \theta_x, \theta_\infty$  and the integration constants than those of [12] or [2]. These solutions, including degenerate cases, are of complex power type, of inverse logarithmic type and of Taylor series type (cf. Theorems 2.1, and 2.3 through 2.5). All of them are derived by a unified method. The key is finding suitable matrix solutions of the Schlesinger equation equivalent to (V) controlling the isomonodromy deformation of (1.1). Solutions of logarithmic type [20] are also obtained from those of inverse logarithmic type through a Bäcklund transformation found by Gromak [4] (cf. Remark 2.7 and Section 6). In such a sense these inverse logarithmic solutions play the same role as that of the basic logarithmic solutions of the sixth Painlevé equation. If  $\theta_0 - \theta_x = \theta_\infty = 0$ , the complex power type of solutions have relatively simple oscillatory expressions although, in general, such expressions are complicated (cf. Theorem 2.2 and Remark 2.4). For the complex power type of solutions in the generic case, we clarify the structure of the analytic continuation on the universal covering around  $x = 0$ , and examine the distribution of zeros, poles and 1-points (note that  $y = 0, 1$  and  $\infty$  are singular values of equation (V)). It is shown that the domains where each solution behaves like a complex power are separated by two kinds of spiral domains including a sector as a special case, which are alternately arrayed; the separating domains of one kind contain sequences both of zeros and of poles, and those of the other kind

sequences of 1-points (cf. Remark 2.9). This situation is different from that of the sixth Painlevé transcendents, in which the separating domain with zeros and that with poles alternately appear (cf. [6], [7], [10], [11], [19]).

All the results above are described in Section 2. In Section 5 the families of solutions of three types are derived from the matrix solutions of the Schlesinger equation given in Section 3 by using the lemmas in Section 4. In Sections 6 and 7 we prove the result on the analytic continuation, those on the distribution of zeros, poles and 1-points, and Theorem 2.2 on the special oscillatory expressions. In proving them, the Bäcklund transformation referred to above is crucial. Section 8 is devoted to the summary of the argument in [12, §2] concerning limiting procedure applied to (1.1) and its monodromy data, and Section 9 to the computation of the connection formulas for the related Whittaker and hypergeometric systems. In non-generic cases of these linear systems it is also possible to compute the monodromy data (cf. Remark 2.12 and Section 9.3). In the final section, using the material above, we derive the results on the monodromy data for our solutions.

Throughout this paper we use the following symbols:

(1) for a ring  $\mathbb{A}$ ,  $M_2(\mathbb{A})$  is the ring of  $2 \times 2$  matrices whose entries are in  $\mathbb{A}$ , and  $GL_2(\mathbb{A}) := \{C \in M_2(\mathbb{A}); C^{-1} \in M_2(\mathbb{A})\}$ ;

(2)  $I, J, \Delta, \Delta_-$  denote the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Delta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Delta_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};$$

(3)  $\mathcal{R}(\mathbb{C} \setminus \{0\})$  denotes the universal covering of  $\mathbb{C} \setminus \{0\}$ ;

(4)  $\mathbb{Q}_\theta := \mathbb{Q}[\theta_0, \theta_x, \theta_\infty]$ ;

(5) for  $A, B \subset \mathbb{C}$ ,  $\text{cl}(A)$  denotes the closure of  $A$ ,  $\text{dist}(A, B)$  the distance between  $A$  and  $B$ ;

(6)  $\psi(x) := \Gamma'(x)/\Gamma(x)$  is the di-Gamma function.

## 2. MAIN RESULTS

**2.1. Solutions near  $x = 0$ .** Set  $\mathbb{Q}_\theta := \mathbb{Q}[\theta_0, \theta_x, \theta_\infty]$ . For each  $(\theta_0, \theta_x, \theta_\infty) \in \mathbb{C}^3$ , we have solutions of complex power type near  $x = 0$ .

**Theorem 2.1.** *Let  $\Sigma_0$  be a bounded domain satisfying*

$$\Sigma_0 \subset \mathbb{C} \setminus \Sigma_* \quad \text{with} \quad \Sigma_* = \{\sigma \leq -1\} \cup \{0\} \cup \{\sigma \geq 1\} \subset \mathbb{R}$$

*and  $\text{dist}(\Sigma_0, \Sigma_*) > 0$ . Suppose that  $(\sigma^2 - (\theta_0 \pm \theta_x)^2)(\sigma^2 - \theta_\infty^2) \neq 0$  for every  $\sigma \in \text{cl}(\Sigma_0)$ . Then (V) admits a two-parameter family of solutions  $\{y(\sigma, \rho, x); (\sigma, \rho) \in \Sigma_0 \times (\mathbb{C} \setminus \{0\})\}$  with the following properties.*

(i)  *$y(\sigma, \rho, x)$  is holomorphic in  $(\sigma, \rho, x) \in \Omega^+(\Sigma_0, \varepsilon_0) \cup \Omega^-(\Sigma_0, \varepsilon_0) \subset \Sigma_0 \times (\mathbb{C} \setminus \{0\}) \times \mathcal{R}(\mathbb{C} \setminus \{0\})$ , where*

$$\Omega^\pm(\Sigma_0, \varepsilon_0) := \bigcup_{(\sigma, \rho) \in \Sigma_0 \times (\mathbb{C} \setminus \{0\})} \{(\sigma, \rho)\} \times \Omega_{\sigma, \rho}^\pm(\varepsilon_0),$$

$$\begin{aligned}\Omega_{\sigma,\rho}^+(\varepsilon_0) &:= \{x \in \mathcal{R}(\mathbb{C} \setminus \{0\}); | \rho x^\sigma | < \varepsilon_0, |x(\rho x^\sigma)^{-1}| < \varepsilon_0\}, \\ \Omega_{\sigma,\rho}^-(\varepsilon_0) &:= \{x \in \mathcal{R}(\mathbb{C} \setminus \{0\}); |x(\rho x^\sigma)| < \varepsilon_0, |(\rho x^\sigma)^{-1}| < \varepsilon_0\},\end{aligned}$$

$\varepsilon_0 = \varepsilon_0(\Sigma_0, \theta_0, \theta_x, \theta_\infty)$  being a sufficiently small number depending only on  $\Sigma_0$  and  $(\theta_0, \theta_x, \theta_\infty)$ .

(ii)  $y(\sigma, \rho, x)$  is represented by the convergent series as follows:

$$\begin{aligned}y_+(\sigma, \rho, x) &:= 1 + \frac{4\sigma^2(\theta_0 + \theta_x - \sigma)}{(\sigma + \theta_\infty)(\theta_0^2 - (\sigma + \theta_x)^2)} \rho x^\sigma \\ &\quad + \sum_{j \geq 2} c_j^+(\sigma) (\rho x^\sigma)^j + \sum_{n=1}^{\infty} x^n \sum_{j \geq 0} c_{jn}^+(\sigma) (\rho x^\sigma)^{-n+j}\end{aligned}$$

in  $\Omega^+(\Sigma_0, \varepsilon_0)$ , and

$$\begin{aligned}y_-(\sigma, \rho, x) &:= 1 - \frac{4\sigma^2(\theta_0 + \theta_x + \sigma)}{(\sigma - \theta_\infty)(\theta_0^2 - (\sigma - \theta_x)^2)} (\rho x^\sigma)^{-1} \\ &\quad + \sum_{j \geq 2} c_j^-(\sigma) (\rho x^\sigma)^{-j} + \sum_{n=1}^{\infty} x^n \sum_{j \geq 0} c_{jn}^-(\sigma) (\rho x^\sigma)^{n-j}\end{aligned}$$

in  $\Omega^-(\Sigma_0, \varepsilon_0)$ , where  $c_j^\pm(\sigma), c_{jn}^\pm(\sigma) \in \mathbb{Q}_\theta(\sigma)$ .

*Remark 2.1.* Note that  $|x| < \varepsilon_0^2$  in  $\Omega_{\sigma,\rho}^\pm(\varepsilon_0)$ . For each  $(\sigma, \rho)$ ,  $\Omega_{\sigma,\rho}^+(\varepsilon_0)$  is given by

$$\begin{aligned}\operatorname{Re} \sigma \cdot \log |x| + \log |\rho| + \log(\varepsilon_0^{-1}) &< \operatorname{Im} \sigma \cdot \arg x \\ &< (\operatorname{Re} \sigma - 1) \log |x| + \log |\rho| - \log(\varepsilon_0^{-1}),\end{aligned}$$

and  $\Omega_{\sigma,\rho}^-(\varepsilon_0)$  by

$$\begin{aligned}(\operatorname{Re} \sigma + 1) \log |x| + \log |\rho| + \log(\varepsilon_0^{-1}) &< \operatorname{Im} \sigma \cdot \arg x \\ &< \operatorname{Re} \sigma \cdot \log |x| + \log |\rho| - \log(\varepsilon_0^{-1}).\end{aligned}$$

*Remark 2.2.* The asymptotic solution with  $0 < \operatorname{Re} \sigma < 1$  in [2, Theorem 6.1] coincides with  $y_+(\sigma, \rho, x)$ .

*Remark 2.3.* For each  $(\sigma, \rho)$ , in the domain

$$\begin{aligned}\Omega_{\sigma,\rho}(\varepsilon_0) &:= \{x \in \mathcal{R}(\mathbb{C} \setminus \{0\}); |x(\rho x^\sigma)| < \varepsilon_0, |x(\rho x^\sigma)^{-1}| < \varepsilon_0\} \\ &\supset \Omega_{\sigma,\rho}^+(\varepsilon_0) \cup \Omega_{\sigma,\rho}^-(\varepsilon_0),\end{aligned}$$

$y(\sigma, \rho, x)$  is meromorphic and represented by the ratio of convergent series (see Sections 5.1 and 7). In a special case, such expressions of  $y(\sigma, \rho, x)$  are relatively simple formulas as in the following theorem, which may be regarded as counterparts of an elliptic representation for the sixth Painlevé transcendents [6], [7].

**Theorem 2.2.** Suppose that  $\theta_0 - \theta_x = \theta_\infty = 0$  and  $\sigma \in \Sigma_0$ .

(1) If  $\sigma^2 - 4\theta_0^2 \neq 0$  for every  $\sigma \in \operatorname{cl}(\Sigma_0)$ , then

$$(2.1) \quad y(\sigma, \rho, x) = \tanh^2 \left( \frac{1}{2} \log(\tilde{\rho} x^\sigma) + \sum_{n=1}^{\infty} x^n \sum_{j=-n}^n c_{jn}(\sigma) (\tilde{\rho} x^\sigma)^j \right)$$

with  $\tilde{\rho} = (2\theta_0 - \sigma)(2\theta_0 + \sigma)^{-1}\rho$ , in which the series with  $c_{jn}(\sigma) \in \mathbb{Q}[\theta_0](\sigma)$  converges in  $\Omega_{\sigma, \tilde{\rho}}(\varepsilon_0)$ .

(2) If  $(\sigma + 1)^2 - (2\theta_0 - 1)^2 \neq 0$  for every  $\sigma \in \text{cl}(\Sigma_0)$ , then

$$(2.2) \quad y(\sigma, \rho, x) = 1 - \frac{2x \sinh(\log(\check{\rho}x^{\sigma+1}) + \Sigma(x))}{2(\sigma + 1) + 2x\Sigma'(x) + x \sinh(\log(\check{\rho}x^{\sigma+1}) + \Sigma(x))}$$

with

$$\check{\rho} = \frac{(\sigma + 2)(1 - \sigma)(2\theta_0 - \sigma)}{8\sigma(\sigma + 1)^2(\sigma + 3)}\rho,$$

where the series

$$\Sigma(x) = \sum_{n=1}^{\infty} x^n \sum_{j=-n}^n \tilde{c}_{jn}(\sigma)(\check{\rho}x^{\sigma+1})^j, \quad \tilde{c}_{jn}(\sigma) \in \mathbb{Q}[\theta_0](\sigma)$$

and  $\Sigma'(x) = (d/dx)\Sigma(x)$  converge in  $\Omega_{\sigma+1, \check{\rho}}(\varepsilon_0)$ .

*Remark 2.4.* The expressions above describe oscillatory behaviours. Indeed, if  $|\tilde{\rho}x^{\sigma}|^{\pm 1}$  are bounded, then (2.1) is

$$-\tan^2(-(i/2)\log(\tilde{\rho}x^{\sigma}) + O(x)) = -\tan^2((1/2)\arg(\tilde{\rho}x^{\sigma}) - (i/2)\log|\tilde{\rho}x^{\sigma}| + O(x)),$$

and, if  $|\check{\rho}x^{\sigma+1}|^{\pm 1}$  are bounded, then (2.2) is

$$1 - ((\sigma + 1)^{-1} + O(x))x \sin(\arg(\check{\rho}x^{\sigma+1}) - i \log|\check{\rho}x^{\sigma+1}| + O(x)).$$

In the case where  $\theta_0 - \theta_x \neq 0$  or  $\theta_{\infty} \neq 0$  as well,  $y(\sigma, \rho, x)$  admits oscillatory expressions as follows (cf. Section 7):

(i) under the suppositions of Theorem 2.1, if  $|\rho x^{\sigma}|^{\pm 1}$  are bounded,

$$y(\sigma, \rho, x) = \frac{\Phi_1(x)\Phi_2(x)F(x)G(x) + O(x)}{\Psi_1(x)\Psi_2(x)F(x)G(x) + O(x)},$$

(ii) under certain generic conditions added to the suppositions of Theorem 2.1, if  $|\hat{\rho}x^{\sigma+1}|^{\pm 1}$  are bounded,

$$1 - \frac{1}{y(\sigma, \rho, x)} = \frac{x(\Phi_1^{\pi}(x)\Phi_2^{\pi}(x)\Psi_1^{\pi}(x)\Psi_2^{\pi}(x)F^{\pi}(x)^2G^{\pi}(x)^2 + O(x))}{2(\sigma + 1)^2((\sigma + 1)^2 - (1 - \theta_0 + \theta_x)^2)\Phi_2^{\pi}(x)\Psi_1^{\pi}(x)F^{\pi}(x)^2G^{\pi}(x)^2 + O(x)}.$$

Here  $\Phi_1(x)$ ,  $\Phi_2(x)$ ,  $\Psi_1(x)$ ,  $\Psi_2(x)$ ,  $F(x)$ ,  $G(x)$  are as in Section 7.1, and, say  $\Phi_1^{\pi}(x)$ , the result of the substitution

$$(\sigma, \rho, \theta_0 - \theta_x, \theta_0 + \theta_x, \theta_{\infty}) \mapsto (\sigma + 1, \hat{\rho}, 1 - \theta_{\infty}, 1 - \theta_0 + \theta_x, \theta_0 + \theta_x - 1)$$

in  $\Phi_1(x)$  (cf. (2.4)),  $\hat{\rho}$  being such that

$$\hat{\rho} = \frac{(\sigma - \theta_{\infty})(2 - \theta_0 + \theta_x + \sigma)(\theta_0 + \theta_x - \sigma)}{8\sigma^2(\sigma + 1)^2}\rho$$

(cf. the proof of Theorem 2.10). In the expressions above it seems that the cancellations of the factors  $F(x)G(x)$ ,  $\Phi_2^{\pi}(x)\Psi_1^{\pi}(x)F^{\pi}(x)^2G^{\pi}(x)^2$  occur, but we have not succeeded in proving them.

For each  $(\theta_0, \theta_x, \theta_{\infty})$  we have solutions of special complex power type.

**Theorem 2.3.** *Suppose that  $\theta_0\theta_x \neq 0$ . Let  $\sigma_0 = \theta_0 \pm \theta_x$  or  $\theta_x - \theta_0$  be such that  $\sigma_0 \in \Sigma_+ := \mathbb{C} \setminus (\{\sigma \leq -1\} \cup \mathbb{Z})$  and  $\sigma_0^2 - \theta_\infty^2 \neq 0$ . Then (V) admits a one-parameter family of solutions  $\{y_{\sigma_0}(\rho, x); \rho \in \mathbb{C}\}$  with the following properties.*

(i)  $y_{\sigma_0}(\rho, x)$  is holomorphic in  $(\rho, x) \in \Omega^0(\varepsilon_0) \cup \Omega^-(\sigma_0, \varepsilon_0) \subset \mathbb{C} \times \mathcal{R}(\mathbb{C} \setminus \{0\})$ , where

$$\begin{aligned}\Omega^0(\varepsilon_0) &:= \bigcup_{\rho \in \mathbb{C}} \{\rho\} \times \Omega_\rho^0(\varepsilon_0), \quad \Omega^-(\sigma_0, \varepsilon_0) := \bigcup_{\rho \in \mathbb{C} \setminus \{0\}} \{\rho\} \times \Omega_{\sigma_0, \rho}^-(\varepsilon_0), \\ \Omega_\rho^0(\varepsilon_0) &:= \{x \in \mathcal{R}(\mathbb{C} \setminus \{0\}); |x| < \varepsilon_0, |\rho x^{\sigma_0}| < \varepsilon_0\}, \\ \Omega_{\sigma_0, \rho}^-(\varepsilon_0) &:= \{x \in \mathcal{R}(\mathbb{C} \setminus \{0\}); |x(\rho x^{\sigma_0})| < \varepsilon_0, |(\rho x^{\sigma_0})^{-1}| < \varepsilon_0\},\end{aligned}$$

$\varepsilon_0 = \varepsilon_0(\theta_0, \theta_x, \theta_\infty)$  being a sufficiently small number depending only on  $(\theta_0, \theta_x, \theta_\infty)$ .

(ii)  $y_{\sigma_0}(\rho, x)$  is represented by the convergent series as follows:

(ii.a) if  $\sigma_0 = \theta_0 + \theta_x$ , then

$$1 - \frac{\sigma_0^2}{\theta_0\theta_x}\rho x^{\sigma_0} + \sum_{j \geq 2} c_j^0(\sigma_0)(\rho x^{\sigma_0})^j + \sum_{n=1}^{\infty} x^n \sum_{j \geq 0} c_{jn}^0(\sigma_0)(\rho x^{\sigma_0})^j$$

in  $\Omega^0(\varepsilon_0)$ , and

$$1 - \frac{4\sigma_0^2}{\sigma_0^2 - \theta_\infty^2}(\rho x^{\sigma_0})^{-1} + \sum_{j \geq 2} \tilde{c}_j^0(\sigma_0)(\rho x^{\sigma_0})^{-j} + \sum_{n=1}^{\infty} x^n \sum_{j \geq 0} \tilde{c}_{jn}^0(\sigma_0)(\rho x^{\sigma_0})^{n-j}$$

in  $\Omega^-(\sigma_0, \varepsilon_0)$ ;

(ii.b) if  $\sigma_0 = \theta_0 - \theta_x$ , then

$$-\frac{\sigma_0 - \theta_\infty}{\sigma_0 + \theta_\infty} \left( 1 - \frac{\sigma_0}{\theta_0} \rho x^{\sigma_0} + \sum_{j \geq 2} c_j^0(\sigma_0)(\rho x^{\sigma_0})^j + \sum_{n=1}^{\infty} x^n \sum_{j \geq 0} c_{jn}^0(\sigma_0)(\rho x^{\sigma_0})^j \right)$$

in  $\Omega^0(\varepsilon_0)$ , and

$$1 - \frac{4\theta_0\sigma_0}{\sigma_0^2 - \theta_\infty^2}(\rho x^{\sigma_0})^{-1} + \sum_{j \geq 2} \tilde{c}_j^0(\sigma_0)(\rho x^{\sigma_0})^{-j} + \sum_{n=1}^{\infty} x^n \sum_{j \geq 0} \tilde{c}_{jn}^0(\sigma_0)(\rho x^{\sigma_0})^{n-j}$$

in  $\Omega^-(\sigma_0, \varepsilon_0)$ ;

(ii.c) if  $\sigma_0 = \theta_x - \theta_0$ , then

$$-\frac{\sigma_0 + \theta_\infty}{\sigma_0 - \theta_\infty} \left( 1 - \frac{\sigma_0}{\theta_x} \rho x^{\sigma_0} + \sum_{j \geq 2} c_j^0(\sigma_0)(\rho x^{\sigma_0})^j + \sum_{n=1}^{\infty} x^n \sum_{j \geq 0} c_{jn}^0(\sigma_0)(\rho x^{\sigma_0})^j \right)$$

in  $\Omega^0(\varepsilon_0)$ , and

$$1 - \frac{4\theta_x\sigma_0}{\sigma_0^2 - \theta_\infty^2}(\rho x^{\sigma_0})^{-1} + \sum_{j \geq 2} \tilde{c}_j^0(\sigma_0)(\rho x^{\sigma_0})^{-j} + \sum_{n=1}^{\infty} x^n \sum_{j \geq 0} \tilde{c}_{jn}^0(\sigma_0)(\rho x^{\sigma_0})^{n-j}$$

in  $\Omega^-(\sigma_0, \varepsilon_0)$ .

Here  $c_j^0(\sigma_0), c_{jn}^0(\sigma_0) \in \mathbb{Q}_\theta[\theta_0^{-1}, \theta_x^{-1}](\sigma_0)$  and  $\tilde{c}_j^0(\sigma_0), \tilde{c}_{jn}^0(\sigma_0) \in \mathbb{Q}_\theta(\sigma_0)$ .

*Remark 2.5.* For each  $\rho \neq 0$ ,  $\Omega_\rho^0(\varepsilon_0)$  is given by

$$|x| < \varepsilon_0, \quad \operatorname{Re} \sigma_0 \cdot \log |x| + \log |\rho| + \log(\varepsilon_0^{-1}) < \operatorname{Im} \sigma_0 \cdot \arg x,$$

and  $\Omega_{\sigma_0, \rho}^-(\varepsilon_0)$  by

$$(1 + \operatorname{Re} \sigma_0) \log |x| + \log |\rho| + \log(\varepsilon_0^{-1}) < \operatorname{Im} \sigma_0 \cdot \arg x \\ < \operatorname{Re} \sigma_0 \cdot \log |x| + \log |\rho| - \log(\varepsilon_0^{-1}).$$

*Remark 2.6.* In  $\Omega_0^0(\varepsilon_0)$ ,  $y_{\sigma_0}(0, x)$  in each case is a Taylor series solution. If  $\sigma_0 = \pm(\theta_0 - \theta_x)$  then  $y_{\sigma_0}(0, x) = -(\theta_0 - \theta_x - \theta_\infty)/(\theta_0 - \theta_x + \theta_\infty) + O(x)$ , and if  $\sigma_0 = \theta_0 + \theta_x$ , direct substitution into (V) yields  $y_{\sigma_0}(0, x) = 1 + (1 - \theta_0 - \theta_x)^{-1}x + O(x^2)$ . Since  $\sigma_0 \notin \mathbb{Z}$ , the coefficients  $c_{0n}^0(\sigma_0)$  of both solutions are uniquely determined, and they coincide with the solutions (II) and (III) in [14, Theorem 2], respectively.

The following are solutions of inverse logarithmic type, which correspond to the Chazy solutions of the sixth Painlevé equation [15].

**Theorem 2.4.** (1) Suppose that  $\theta_\infty \neq 0$  and  $\theta_0^2 - \theta_x^2 \neq 0$ . Then (V) admits a one-parameter family of solutions  $\{y_{\text{ilog}}(\rho, x); \rho \in \mathcal{R}(\mathbb{C} \setminus \{0\})\}$  such that  $y_{\text{ilog}}(\rho, x)$  is holomorphic in  $(\rho, x) \in \Omega^*(\varepsilon_0, \Theta_0) \subset \mathcal{R}(\mathbb{C} \setminus \{0\})^2$  and is represented by the convergent series

$$y_{\text{ilog}}(\rho, x) = 1 + \frac{4}{\theta_\infty(\theta_0 - \theta_x)} \log^{-2}(\rho x) + \sum_{j \geq 3} c_j \log^{-j}(\rho x) + \sum_{n=1}^{\infty} x^n \sum_{j \geq 0} c_{jn} \log^{2n-j}(\rho x)$$

with  $c_j, c_{jn} \in \mathbb{Q}_\theta[\theta_\infty^{-1}, (\theta_0^2 - \theta_x^2)^{-1}]$ , where

$$\Omega^*(\varepsilon_0, \Theta_0) := \bigcup_{\rho \in \mathcal{R}(\mathbb{C} \setminus \{0\})} \{\rho\} \times \Omega_\rho^*(\varepsilon_0, \Theta_0),$$

$$\Omega_\rho^*(\varepsilon_0, \Theta_0) := \{x \in \mathcal{R}(\mathbb{C} \setminus \{0\}); |\rho x| < \varepsilon_0, |x(\rho x)^{-1/2}| < \varepsilon_0, |\arg(\rho x)| < \Theta_0\},$$

$\Theta_0$  being a given positive number and  $\varepsilon_0 = \varepsilon_0(\Theta_0, \theta_0, \theta_x, \theta_\infty)$  a sufficiently small number depending only on  $\Theta_0$  and  $(\theta_0, \theta_x, \theta_\infty)$ .

(2) Suppose that  $\theta_\infty \neq 0$  and  $\theta_0^2 - \theta_x^2 = 0$ . If  $\theta_0 = \theta_x \neq 0$  or if  $\theta_0 = -\theta_x \neq 0$ , then (V) admits one-parameter families of solutions  $\{y_{\text{ilog}}^\pm(\rho, x); \rho \in \mathcal{R}(\mathbb{C} \setminus \{0\})\}$  or  $\{y_{\text{ilog}}^+(\rho, x); \rho \in \mathcal{R}(\mathbb{C} \setminus \{0\})\}$ , respectively, such that each solution is holomorphic in  $(\rho, x) \in \Omega^*(\varepsilon_0, \Theta_0)$  and is represented by the convergent series as follows:

(i) if  $\theta_0 = \theta_x \neq 0$ ,

$$y_{\text{ilog}}^\pm(\rho, x) = 1 \mp \frac{2}{\theta_\infty} \log^{-1}(\rho x) + \sum_{j \geq 2} c_j^\pm \log^{-j}(\rho x) + \sum_{n=1}^{\infty} x^n \sum_{j \geq 0} c_{jn}^\pm \log^{n-j}(\rho x);$$

(ii) if  $\theta_0 = -\theta_x \neq 0$ ,

$$y_{\text{ilog}}^+(\rho, x) = 1 + \frac{2}{\theta_0 \theta_\infty} \log^{-2}(\rho x) + \sum_{j \geq 3} c_j^+ \log^{-j}(\rho x) + \sum_{n=1}^{\infty} x^n \sum_{j \geq 0} c_{jn}^+ \log^{n-j}(\rho x).$$

Here  $c_j^\pm, c_{jn}^\pm \in \mathbb{Q}[\theta_0, \theta_\infty^{-1}, \theta_\infty, \theta_\infty^{-1}]$ , in particular, in case  $\theta_0 = \theta_x$ ,  $c_j^+ = 0$  for  $j \geq 2$ .

(3) Suppose that  $\theta_\infty = 0$ . If  $\theta_0^2 - \theta_x^2 \neq 0$  or if  $\theta_0 = -\theta_x \neq 0$ , then (V) admits one-parameter families of solutions  $\{y_{\text{ilog}}^{(l)}(\rho, x); \rho \in \mathcal{R}(\mathbb{C} \setminus \{0\})\}$  ( $l = 1, 2$ ) or  $\{y_{\text{ilog}}^{(1)}(\rho, x); \rho \in \mathcal{R}(\mathbb{C} \setminus \{0\})\}$ , respectively, such that each solution is holomorphic in  $(\rho, x) \in \Omega^*(\varepsilon_0, \Theta_0)$  and is represented by the convergent series as follows:

(i) if  $\theta_0^2 - \theta_x^2 \neq 0$ ,

$$y_{\text{ilog}}^{(l)}(\rho, x) = 1 + \frac{(-1)^{l+1}2}{\theta_0 - \theta_x} \log^{-1}(\rho x) + \sum_{j \geq 2} c_j^{(l)} \log^{-j}(\rho x) + \sum_{n=1}^{\infty} x^n \sum_{j \geq 0} c_{jn}^{(l)} \log^{2n-j}(\rho x);$$

(ii) if  $\theta_0 = -\theta_x \neq 0$ ,

$$y_{\text{ilog}}^{(1)}(\rho, x) = 1 + \frac{1}{\theta_0} \log^{-1}(\rho x) + \sum_{j \geq 2} c_j^{(1)} \log^{-j}(\rho x) + \sum_{n=1}^{\infty} x^n \sum_{j \geq 0} c_{jn}^{(1)} \log^{n-j}(\rho x).$$

Here  $c_j^{(l)}, c_{jn}^{(l)} \in \mathbb{Q}[\theta_0, \theta_x, (\theta_0^2 - \theta_x^2)^{-1}]$  ( $l = 1, 2$ ) and  $c_j^{(1)}, c_{jn}^{(1)} \in \mathbb{Q}[\theta_0, \theta_0^{-1}]$ , respectively.

*Remark 2.7.* Applying the Bäcklund transformation with  $\pi$  in Lemma 6.1 to the inverse logarithmic solutions above except for  $y_{\text{ilog}}^+(\rho, x)$  with  $\theta_0 = \theta_x$ , we derive solutions of logarithmic type as follows: under the condition  $\theta_0 + \theta_x \neq 1$ , solutions satisfying

$$y_{\text{log}}(\rho, x) \sim 1 - \frac{1}{2}(1 - \theta_0 - \theta_x)x \log^2(\rho x)$$

with  $(1 - \theta_\infty)(1 - \theta_0 + \theta_x) \neq 0$ , with  $\theta_\infty = 1, \theta_0 - \theta_x \neq 1$  and with  $\theta_\infty \neq 1, \theta_0 - \theta_x = 1$  follow from  $y_{\text{ilog}}(\rho, x)$ , from  $y_{\text{ilog}}^-(\rho, x)$  with  $\theta_0 = \theta_x \neq 0$  and from  $y_{\text{ilog}}^+(\rho, x)$  with  $\theta_0 = -\theta_x \neq 0$ , respectively; and under the condition  $\theta_0 + \theta_x = 1$ , those satisfying

$$y_{\text{log}}^{(l)}(\rho, x) \sim 1 + (-1)^{l+1}x \log(\rho x) \quad (l = 1, 2)$$

with  $(1 - \theta_\infty)(1 - \theta_0 + \theta_x) \neq 0$  and with  $\theta_\infty \neq 1, \theta_0 - \theta_x = 1$  follow from  $y_{\text{ilog}}^{(l)}(\rho, x)$  ( $l = 1, 2$ ) with  $\theta_0^2 - \theta_x^2 \neq 0$  and from  $y_{\text{ilog}}^{(1)}(\rho, x)$  with  $\theta_0 = -\theta_x \neq 0$ , respectively. These logarithmic solutions are studied in [20]. For the exceptional case of  $y_{\text{ilog}}^+(\rho, x)$  the denominator of the Bäcklund transformation is of the form  $x(\cdots)$ , and to compute the resultant solution we need to know some of the coefficients  $c_{jn}^+$ .

**Theorem 2.5.** Suppose that  $\theta_\infty = 0$ . If  $\theta_0 = \theta_x$  or if  $\theta_0 = -\theta_x$ , then (V) has a one-parameter family of solutions  $\{y_{\text{Taylor}}^+(a, x); a \in \mathbb{C} \setminus \{0\}\}$  or  $\{y_{\text{Taylor}}^-(a, x); a \in \mathbb{C}\}$ , respectively, represented by the convergent series

$$y_{\text{Taylor}}^\pm(a, x) = \sum_{n=0}^{\infty} c_n^\pm(a) x^n$$

with  $c_0^+(a) = (a + \theta_0)/a$ ,  $c_1^+(a) = (a + \theta_0)(1 - 2\theta_0)/a$ ,  $c_n^+(a) \in \mathbb{Q}[a, a^{-1}, \theta_0]$ , or  $c_0^-(a) = c_1^-(a) = 1$ ,  $c_2^-(a) = (1 - \theta_0 - 2a)/2$ ,  $c_n^-(a) \in \mathbb{Q}[a, \theta_0]$ . If  $\theta_0 = \theta_x = 0$ , then  $c_n^+(a) = c_n^-(a)$  for every  $n \geq 0$ .



**2.2. Analytic continuation.** Suppose that  $(\sigma^2 - (\theta_0 \pm \theta_x)^2)(\sigma^2 - \theta_\infty^2) \neq 0$  for every  $\sigma \in \text{cl}(\Sigma_0)$ . Let us discuss the analytic continuation of  $y(\sigma, \rho, x)$  on  $\mathcal{R}(\mathbb{C} \setminus \{0\})$  around  $x = 0$ . If  $\sigma \in \Sigma_0$  satisfies  $0 < \sigma < 1$  (respectively,  $-1 < \sigma < 0$ ), then the solution  $y_+(\sigma, \rho, x)$  (respectively,  $y_-(\sigma, \rho, x)$ ) in Theorem 2.1 converges in  $\{x \in \mathcal{R}(\mathbb{C} \setminus \{0\}); |x| < \varepsilon'_0\}$  for some  $\varepsilon'_0 > 0$  (in fact  $y_+(\sigma, \rho, x) \equiv y_-(-\sigma, \rho^{-1}, x)$ ).

In what follows suppose that  $(\sigma, \rho) \in \Sigma_0 \times (\mathbb{C} \setminus \{0\})$  satisfies  $\text{Im } \sigma \neq 0$ . For  $\nu \in \mathbb{Z}$  let  $D_\pm(\sigma, \rho, \nu) \subset \mathcal{R}(\mathbb{C} \setminus \{0\})$  be domains given by

$$\begin{aligned} D_+(\sigma, \rho, \nu) : \quad & (\text{Re } \sigma - 2\nu) \log |x| + \log |\rho| + \log(\varepsilon_0^{-1}) < \text{Im } \sigma \cdot \arg x \\ & < (\text{Re } \sigma - 2\nu - 1) \log |x| + \log |\rho| - \log(\varepsilon_0^{-1}), \\ D_-(\sigma, \rho, \nu) : \quad & (\text{Re } \sigma - 2\nu + 1) \log |x| + \log |\rho| + \log(\varepsilon_0^{-1}) < \text{Im } \sigma \cdot \arg x \\ & < (\text{Re } \sigma - 2\nu) \log |x| + \log |\rho| - \log(\varepsilon_0^{-1}). \end{aligned}$$

Furthermore set

$$\begin{aligned} D_{\text{even}}(\sigma, \rho, \nu) : \quad & |x| < \varepsilon_0, \\ & -\log(\varepsilon_0^{-1}) < (\text{Re } \sigma - 2\nu) \log |x| - \text{Im } \sigma \cdot \arg x + \log |\rho| < \log(\varepsilon_0^{-1}), \\ D_{\text{odd}}(\sigma, \rho, \nu) : \quad & |x| < \varepsilon_0, \\ & -\log(\varepsilon_0^{-1}) < (\text{Re } \sigma - 2\nu + 1) \log |x| - \text{Im } \sigma \cdot \arg x + \log |\rho| < \log(\varepsilon_0^{-1}). \end{aligned}$$

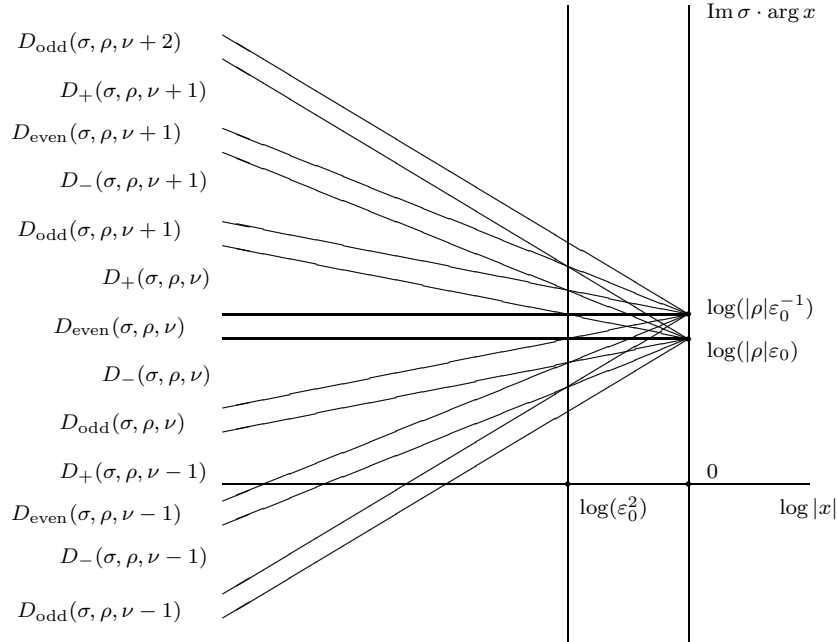


Figure 2.1.

In general these are spiral domains, and

$$\bigcup_{\nu \in \mathbb{Z}} \left( \text{cl}(D_{\text{odd}}(\sigma, \rho, \nu)) \cup D_-(\sigma, \rho, \nu) \cup \text{cl}(D_{\text{even}}(\sigma, \rho, \nu)) \cup D_+(\sigma, \rho, \nu) \right)$$

contains  $\{x \in \mathcal{R}(\mathbb{C} \setminus \{0\}); |x| < \varepsilon_0^2\}$ . For every  $\nu \in \mathbb{Z}$ , by Theorem 2.1,  $y(\sigma - 2\nu, \rho, x)$  with  $\sigma \in \Sigma_0$  is represented by  $y_+(\sigma - 2\nu, \rho, x)$  in  $D_+(\sigma, \rho, \nu)$  and by  $y_-(\sigma - 2\nu, \rho, x)$  in  $D_-(\sigma, \rho, \nu)$ , as long as  $((\sigma - 2\nu)^2 - (\theta_0 \pm \theta_x)^2)((\sigma - 2\nu)^2 - \theta_\infty^2) \neq 0$  for  $\sigma \in \text{cl}(\Sigma_0)$ . Set

$$(2.3) \quad c(\sigma) := \frac{4\sigma^2(\theta_0 + \theta_x - \sigma)}{(\sigma + \theta_\infty)(\theta_0^2 - (\sigma + \theta_x)^2)}.$$

To  $c(\sigma)$  apply the substitution

$$(2.4) \quad \pi : (\theta_0 - \theta_x, \theta_0 + \theta_x, \theta_\infty) \mapsto (1 - \theta_\infty, 1 - \theta_0 + \theta_x, \theta_0 + \theta_x - 1)$$

and denote the result by  $c^\pi(\sigma) = \tilde{c}(\sigma)$ . Then we have the following relation, which gives the analytic continuation of  $y(\sigma, \rho, x)$  on  $\{x \in \mathcal{R}(\mathbb{C} \setminus \{0\}); |x| < \varepsilon_0^2\}$ .

**Theorem 2.6.** *For every  $\nu \in \mathbb{Z}$ ,*

$$y(\sigma - 2\nu + 2, \rho, x) \equiv y(\sigma - 2\nu, \gamma(\sigma, \nu)\rho, x)$$

with

$$\begin{aligned} \gamma(\sigma, \nu) &= \frac{1}{4}(2\nu - \theta_\infty - \sigma)(\sigma - \theta_\infty - 2\nu + 2) \\ &\quad \times c(2\nu - \sigma)c(\sigma - 2\nu + 2)\tilde{c}(\sigma - 2\nu + 1)\tilde{c}(2\nu - 1 - \sigma) \\ &= \frac{64(\sigma - 2\nu)^2(\sigma - 2\nu + 1)^4(\sigma - 2\nu + 2)^2}{(\theta_\infty - \sigma + 2\nu)(\theta_\infty + \sigma - 2\nu + 2)((\sigma - 2\nu - \theta_x)^2 - \theta_0^2)((\sigma - 2\nu + 2 + \theta_x)^2 - \theta_0^2)} \end{aligned}$$

as long as  $\gamma(\sigma, \nu) \neq 0, \infty$  for  $\sigma \in \text{cl}(\Sigma_0)$ .

**2.3. Distribution of poles, zeros and 1-points.** If  $\theta_0 - \theta_x = \theta_\infty = 0$  and  $\text{Im } \sigma \neq 0$ , then, by Remark 2.4, (2.1) in  $D_{\text{even}}(\sigma, \tilde{\rho}, 0)$  has sequences of zeros and of poles lying asymptotically along the curve  $\log |\tilde{\rho}x^\sigma| = \text{Re } \sigma \cdot \log |x| - \text{Im } \sigma \cdot \arg x + \log |\tilde{\rho}| = 0$ . Similarly, (2.2) in  $D_{\text{odd}}(\sigma, \check{\rho}, 0)$  has sequences of 1-points lying asymptotically along the curve  $\log |\check{\rho}x^{\sigma+1}| = (\text{Re } \sigma + 1) \cdot \log |x| - \text{Im } \sigma \cdot \arg x + \log |\check{\rho}| = 0$ . These facts are generalised by the following theorems, in which

$$(2.5) \quad L(r_0, \omega)_\sigma : (1 + \text{Re } \sigma - \omega) \log |x| - \text{Im } \sigma \cdot \arg x = r_0, \quad r_0, \omega \in \mathbb{R}$$

is a spiral curve if  $\text{Re } \sigma \neq \omega - 1$  or a ray if  $\text{Re } \sigma = \omega - 1$ .

**Theorem 2.7.** *In addition to  $(\sigma^2 - \theta_\infty^2)(\sigma^2 - (\theta_0 \pm \theta_x)^2) \neq 0$ , suppose that*

$$\theta_x(\theta_0 + \theta_x - \theta_\infty)(\theta_0^2 - \theta_x^2 + \sigma^2 - 2\theta_0\theta_\infty) \neq 0$$

for  $\sigma \in \text{cl}(\Sigma_0)$  and

$$(2.6) \quad \theta_0 - \theta_x - \theta_\infty \neq 0.$$

Set

$$r_0 := \log |\xi_0|, \quad \mu_0 := \arg \xi_0 \quad \text{with} \quad \xi_0 := -\frac{\sigma + \theta_0 + \theta_x}{\sigma - \theta_0 - \theta_x} \rho^{-1}.$$

Then  $y(\sigma, \rho, x)$  admits a sequence of simple zeros  $\{x_n^0\}_{n \in \mathbb{N}} \subset D_{\text{even}}(\sigma, \rho, 0)$  such that

$$|\sigma|^2 \log |x_n^0| - r_0 \text{Re } \sigma - \mu_0 \text{Im } \sigma \sim -2\pi n |\text{Im } \sigma|$$

and  $\text{dist}(x_n^0, L(r_0, 1)_\sigma) = O(|x_n^0|^2)$ . Furthermore for

$$\hat{\xi}_0 := -\frac{(\sigma + \theta_\infty)((\sigma + \theta_x)^2 - \theta_0^2)}{(\sigma - \theta_\infty)((\sigma - \theta_x)^2 - \theta_0^2)}\rho^{-1}$$

there exists another sequence of simple zeros  $\{\hat{x}_n^0\}_{n \in \mathbb{N}} \subset D_{\text{even}}(\sigma, \rho, 0)$  with similar properties, that is,

$$|\sigma|^2 \log |\hat{x}_n^0| - \hat{r}_0 \text{Re } \sigma - \hat{\mu}_0 \text{Im } \sigma \sim -2\pi n |\text{Im } \sigma|$$

and  $\text{dist}(\hat{x}_n^0, L(\hat{r}_0, 1)_\sigma) = O(|\hat{x}_n^0|^2)$ , where  $\hat{r}_0 := \log |\hat{\xi}_0|$ ,  $\hat{\mu}_0 := \arg \hat{\xi}_0$ .

**Theorem 2.8.** In addition to  $(\sigma^2 - \theta_\infty^2)(\sigma^2 - (\theta_0 \pm \theta_x)^2) \neq 0$ , suppose that

$$\theta_0(\theta_0 + \theta_x - \theta_\infty)(\theta_0^2 - \theta_x^2 - \sigma^2 + 2\theta_x\theta_\infty) \neq 0$$

for  $\sigma \in \text{cl}(\Sigma_0)$  and

$$(2.7) \quad \theta_0 - \theta_x + \theta_\infty \neq 0.$$

Set

$$r_\infty := \log |\xi_\infty|, \quad \mu_\infty := \arg \xi_\infty \quad \text{with} \quad \xi_\infty := \frac{(\sigma + \theta_x)^2 - \theta_0^2}{(\sigma - \theta_x)^2 - \theta_0^2}\rho^{-1}.$$

Then  $y(\sigma, \rho, x)$  admits a sequence of simple poles  $\{x_n^\infty\}_{n \in \mathbb{N}} \subset D_{\text{even}}(\sigma, \rho, 0)$  such that

$$|\sigma|^2 \log |x_n^\infty| - r_\infty \text{Re } \sigma - \mu_\infty \text{Im } \sigma \sim -2\pi n |\text{Im } \sigma|$$

and  $\text{dist}(x_n^\infty, L(r_\infty, 1)_\sigma) = O(|x_n^\infty|^2)$ . Another similar sequence of simple poles  $\{\hat{x}_n^\infty\}_{n \in \mathbb{N}} \subset D_{\text{even}}(\sigma, \rho, 0)$  exists for

$$\hat{\xi}_\infty := \frac{(\sigma + \theta_\infty)(\sigma + \theta_0 + \theta_x)}{(\sigma - \theta_\infty)(\sigma - \theta_0 - \theta_x)}\rho^{-1}.$$

**Theorem 2.9.** If  $\theta_0 - \theta_x - \theta_\infty = 0$  in place of (2.6), then  $\{x_n^0\}_{n \in \mathbb{N}} = \{\hat{x}_n^0\}_{n \in \mathbb{N}}$  is a sequence of double zeros. If  $\theta_0 - \theta_x + \theta_\infty = 0$  in place of (2.7), then  $\{x_n^\infty\}_{n \in \mathbb{N}} = \{\hat{x}_n^\infty\}_{n \in \mathbb{N}}$  is a sequence of double poles.

Note that the singular values of (V) are  $y = 0, 1, \infty$ . The results above describe sequences of zeros and of poles in  $D_{\text{even}}(\sigma, \rho, 0)$ . In  $D_{\text{odd}}(\sigma, \rho, 0)$  there exist sequences of 1-points.

**Theorem 2.10.** In addition to  $(\sigma^2 - \theta_\infty^2)(\sigma^2 - (\theta_0 \pm \theta_x)^2) \neq 0$ , suppose that

$$(2.8) \quad (\theta_0 - 1)(\theta_0 + \theta_x - \theta_\infty)((2 - \theta_0 - \theta_\infty)^2 - \theta_x^2)((\sigma + 1)^2 - (\theta_0 \pm \theta_x - 1)^2) \\ \times ((\sigma + 1)^2 - (1 - \theta_\infty)^2)((\sigma + 1)^2 + (1 - \theta_0)^2 + 2(1 - \theta_\infty)(1 - \theta_0) - \theta_x^2) \neq 0$$

for  $\sigma \in \text{cl}(\Sigma_0)$ . Set

$$r_1 := \log |\xi_1|, \quad \mu_1 := \arg \xi_1$$

with

$$\xi_1 := \frac{(\sigma + 2 - \theta_0 + \theta_x)(\sigma + \theta_\infty)c(-\sigma)\tilde{c}(\sigma + 1)}{2(\sigma + \theta_0 - \theta_x)}\rho^{-1} = \frac{8\sigma^2(\sigma + 1)^2}{(\theta_\infty - \sigma)(\theta_0^2 - (\theta_x - \sigma)^2)}\rho^{-1}.$$

Then  $y(\sigma, \rho, x)$  admits a sequence of simple 1-points  $\{x_n^1\}_{n \in \mathbb{N}} \subset D_{\text{odd}}(\sigma, \rho, 0)$  such that

$$|\sigma + 1|^2 \log |x_n^1| - r_1(\operatorname{Re} \sigma + 1) - \mu_1 \operatorname{Im} \sigma \sim -2\pi n |\operatorname{Im} \sigma|$$

and  $\operatorname{dist}(x_n^1, L(r_1, 0)_\sigma) = O(|x_n^1|^2)$ . Another similar sequence of simple 1-points  $\{\hat{x}_n^1\}_{n \in \mathbb{N}} \subset D_{\text{odd}}(\sigma, \rho, 0)$  exists for

$$\hat{\xi}_1 := -\frac{\sigma + \theta_0 + \theta_x}{\sigma + 2 - \theta_0 - \theta_x} \xi_1.$$

*Remark 2.8.* In the proof of Theorem 2.10 in Section 7, if we use (6.2) instead of (6.1), we obtain a sequence of simple 1-points such that

$$\xi_1 := -\frac{2(\sigma - \theta_0 + \theta_x)}{(\sigma - 2 + \theta_0 - \theta_x)(\sigma - \theta_\infty)c(\sigma)\tilde{c}(1 - \sigma)}\rho^{-1} = \frac{(\theta_\infty + \sigma)(\theta_0^2 - (\theta_x + \sigma)^2)}{8\sigma^2(\sigma - 1)^2}\rho^{-1},$$

$$|\sigma - 1|^2 \log |x_n^1| - r_1(\operatorname{Re} \sigma - 1) - \mu_1 \operatorname{Im} \sigma \sim -2\pi n |\operatorname{Im} \sigma|$$

and  $\operatorname{dist}(x_n^1, L(r_1, 2)_\sigma) = O(|x_n^1|^2)$ , and a similar sequence for

$$\hat{\xi}_1 := \frac{\sigma - 2 + \theta_0 + \theta_x}{\sigma - \theta_0 - \theta_x} \xi_1.$$

*Remark 2.9.* Using Theorems 2.7 through 2.10 combined with the relation of Theorem 2.6, we may find sequences of zeros, of poles and of 1-points beyond  $\operatorname{cl}(D_{\text{odd}}(\sigma, \rho, 0)) \cup D_-(\sigma, \rho, 0) \cup \operatorname{cl}(D_{\text{even}}(\sigma, \rho, 0)) \cup D_+(\sigma, \rho, 0)$ . As a result  $D_{\text{even}}(\sigma, \rho, \nu)$  contains sequences of zeros and of poles, and  $D_{\text{odd}}(\sigma, \rho, \nu)$  those of 1-points. They are lying asymptotically along the respective spiral curves or rays.

For the solutions of special complex power type we have

**Theorem 2.11.** *Under the same supposition as in Theorem 2.3 with  $\rho \neq 0$ , set*

$$r_0^* := \log |\xi_0^*|, \quad \mu_0^* := \arg \xi_0^* \quad \text{with} \quad \xi_0^* := \frac{2\theta_0}{\sigma_0 + \theta_\infty} \rho^{-1},$$

and

$$\hat{\xi}_0^* := \frac{2\theta_x}{\sigma_0 - \theta_\infty} \rho^{-1}, \quad \xi_\infty^* := -\frac{2\theta_x}{\sigma_0 + \theta_\infty} \rho^{-1}, \quad \hat{\xi}_\infty^* := -\frac{2\theta_0}{\sigma_0 - \theta_\infty} \rho^{-1}.$$

(1) *Let  $\sigma_0 = \theta_0 + \theta_x$ . Then, under (2.6),  $y_{\sigma_0}(\rho, x)$  admits a sequence of simple zeros  $\{x_n^{*0}\}_{n \in \mathbb{N}} \subset D_{\text{even}}(\sigma_0, \rho, 0)$  such that*

$$|\sigma_0|^2 \log |x_n^{*0}| - r_0^* \operatorname{Re} \sigma_0 - \mu_0^* \operatorname{Im} \sigma_0 \sim -2\pi n |\operatorname{Im} \sigma_0|$$

and  $\operatorname{dist}(x_n^{*0}, L(r_0^*, 1)_{\sigma_0}) = O(|x_n^{*0}|^2)$ , and a similar sequence of simple zeros  $\{\hat{x}_n^{*0}\}_{n \in \mathbb{N}} \subset D_{\text{even}}(\sigma_0, \rho, 0)$  for  $\hat{\xi}_0^*$ . Under (2.7), there exist sequences of simple poles  $\{x_n^{*\infty}\}_{n \in \mathbb{N}}$ ,  $\{\hat{x}_n^{*\infty}\}_{n \in \mathbb{N}} \subset D_{\text{even}}(\sigma_0, \rho, 0)$  for  $\xi_\infty^*$  and  $\hat{\xi}_\infty^*$ , respectively. If  $\theta_0 - \theta_x - \theta_\infty = 0$ , then  $x_n^{*0} = \hat{x}_n^{*0}$  ( $n \in \mathbb{N}$ ) are double zeros; and if  $\theta_0 - \theta_x + \theta_\infty = 0$ , then  $x_n^{*\infty} = \hat{x}_n^{*\infty}$  ( $n \in \mathbb{N}$ ) are double poles.

(2) *If  $\sigma_0 = \theta_0 - \theta_x$  and  $\theta_0 + \theta_x \neq \theta_\infty$ , then  $y_{\sigma_0}(\rho, x)$  admits sequences of simple zeros  $\{x_n^{*0}\}_{n \in \mathbb{N}}$  and of simple poles  $\{\hat{x}_n^{*\infty}\}_{n \in \mathbb{N}}$  as in (1).*

(3) *If  $\sigma_0 = \theta_x - \theta_0$  and  $\theta_0 + \theta_x \neq \theta_\infty$ , then  $y_{\sigma_0}(\rho, x)$  admits sequences of simple zeros  $\{\hat{x}_n^{*0}\}_{n \in \mathbb{N}}$  and of simple poles  $\{x_n^{*\infty}\}_{n \in \mathbb{N}}$  as in (1).*

**Theorem 2.12.** *In addition to the supposition of Theorem 2.3, suppose (2.8) with  $\sigma = \sigma_0$ . Set*

$$r_1^* := \log |\xi_1^*|, \quad \mu_1^* := \arg \xi_1^* \quad \text{with} \quad \xi_1^* := \frac{2(\sigma_0 + 1)^2 c^*(\sigma_0)}{\sigma_0 + \theta_0 + \theta_x} \rho^{-1},$$

where

$$c^*(\sigma_0) := \begin{cases} -\frac{4\sigma_0^2}{\sigma_0^2 - \theta_\infty^2} & \text{if } \sigma_0 = \theta_0 + \theta_x, \\ -\frac{4\theta_0\sigma_0}{\sigma_0^2 - \theta_\infty^2} & \text{if } \sigma_0 = \theta_0 - \theta_x, \\ -\frac{4\theta_x\sigma_0}{\sigma_0^2 - \theta_\infty^2} & \text{if } \sigma_0 = \theta_x - \theta_0. \end{cases}$$

Then  $y_{\sigma_0}(\rho, x)$  admits a sequence of simple 1-points  $\{x_n^{*1}\}_{n \in \mathbb{N}} \subset D_{\text{odd}}(\sigma_0, \rho, 0)$  such that

$$|\sigma_0 + 1|^2 \log |x_n^{*1}| - r_1^*(\operatorname{Re} \sigma_0 + 1) - \mu_1^* \operatorname{Im} \sigma_0 \sim -2\pi n |\operatorname{Im} \sigma_0|$$

and  $\operatorname{dist}(x_n^{*1}, L(r_1^*, 0)_{\sigma_0}) = O(|x_n^{*1}|^2)$ . Another similar sequence of simple 1-points  $\{\hat{x}_n^{*1}\}_{n \in \mathbb{N}} \subset D_{\text{odd}}(\sigma_0, \rho, 0)$  exists for

$$\hat{\xi}_1^* := -\frac{\sigma_0 + \theta_0 + \theta_x}{\sigma_0 + 2 - \theta_0 - \theta_x} \xi_1^*.$$

*Remark 2.10.* The author believes that, in  $D_{\text{even}}(\cdots)$  and  $D_{\text{odd}}(\cdots)$ , there exists no sequence of zeros, of poles or of 1-points other than those given in Theorems 2.7 through 2.12. Indeed, solution (2.1) with the double zeros and poles given by Theorem 2.9 (respectively, (2.2) with the 1-points given by Theorem 2.10) has no zeros, poles or 1-points other than them in  $D_{\text{even}}(\sigma, \rho, 0)$  (respectively, in  $D_{\text{odd}}(\sigma, \rho, 0)$ ). However we have not succeeded in excluding the possibility of its existence with the exception of the cases of  $y_{\sigma_0}(\rho, x)$  with  $\sigma_0 = \theta_0 + \theta_x$  in  $D_{\text{even}}(\sigma_0, \rho, 0)$  and of  $y(\sigma, \rho, x)$  with  $\theta_0 - \theta_x = \theta_\infty = 0$ .

**2.4. Monodromy data.** Linear system (1.1) given by

$$\frac{dY}{d\lambda} = \left( \frac{A_0(x)}{\lambda} + \frac{A_x(x)}{\lambda - x} + \frac{J}{2} \right) Y$$

with the properties (a) and (b) admits a fundamental matrix solution of the form

$$(2.9) \quad Y(\lambda, x) = (I + O(\lambda^{-1})) e^{(\lambda/2)J} \lambda^{-(\theta_\infty/2)J}$$

as  $\lambda \rightarrow \infty$  through the sector  $-\pi/2 < \arg \lambda < 3\pi/2$ . Other matrix solutions  $Y_1(\lambda, x)$ ,  $Y_2(\lambda, x)$  with the same asymptotic representation through the sectors  $-3\pi/2 < \arg \lambda < \pi/2$ ,  $\pi/2 < \arg \lambda < 5\pi/2$ , respectively, are related to  $Y(\lambda, x)$  by

$$Y(\lambda, x) = Y_1(\lambda, x)S_1, \quad Y_2(\lambda, x) = Y(\lambda, x)S_2,$$

where  $S_1 = I + s_1\Delta_-$  and  $S_2 = I + s_2\Delta$  are Stokes multipliers. Let  $M_0$ ,  $M_x$ ,  $M_\infty$  be monodromy matrices with respect to  $Y(\lambda, x)$  such that  $M_0$ ,  $M_x$ ,  $M_\infty$  are given by loops surrounding  $\lambda = 0$ ,  $x$ ,  $\infty$ , respectively, in the positive sense, and that  $M_\infty M_x M_0 = I$ .

Then the isomonodromy deformation of (1.1) preserving the monodromy data  $M_0, M_x, M_\infty, S_1, S_2$  under a small change of  $x$ , is controlled by the Schlesinger equation

$$(2.10) \quad x \frac{dA_0}{dx} = [A_x, A_0], \quad x \frac{dA_x}{dx} = [A_0, A_x] + \frac{x}{2} [J, A_x]$$

that is equivalent to (V) through

$$(2.11) \quad y(x) = \frac{A_x(x)_{12}(A_0(x)_{11} + \theta_0/2)}{A_0(x)_{12}(A_x(x)_{11} + \theta_x/2)}$$

(for details see [13]).

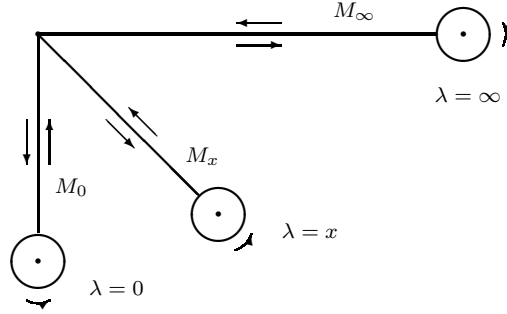


Figure 2.2

The following results give the monodromy data  $M_0, M_x$  related to each solution of (V) by the correspondence described above. Note that  $S_1$  and  $S_2$  follow from the relation  $M_\infty = M_0^{-1} M_x^{-1} = S_2 e^{\pi i \theta_\infty J} S_1$ .

**Theorem 2.13.** *Suppose that  $\theta_0, \theta_x \notin \mathbb{Z}$ . Then the monodromy data for  $y(\sigma, \rho, x)$  of Theorem 2.1 are given by*

$$M_0 = (C_0 C_\infty)^{-1} e^{\pi i \theta_0 J} C_0 C_\infty, \quad M_x = (C_x C_\infty)^{-1} e^{\pi i \theta_x J} C_x C_\infty,$$

where

$$C_\infty = \begin{pmatrix} -\frac{e^{-\pi i(\sigma+\theta_\infty)/2} \Gamma(-\sigma)}{\Gamma(1-(\sigma-\theta_\infty)/2)} & -\frac{\Gamma(-\sigma)}{\Gamma(1-(\sigma+\theta_\infty)/2)} \\ -\frac{e^{\pi i(\sigma-\theta_\infty)/2} \Gamma(\sigma)}{\Gamma((\sigma+\theta_\infty)/2)} & \frac{\Gamma(\sigma)}{\Gamma((\sigma-\theta_\infty)/2)} \end{pmatrix},$$

$$C_0 = \tilde{C}_0 \begin{pmatrix} 1 & 0 \\ 0 & 2\rho/(\sigma+\theta_\infty) \end{pmatrix}, \quad C_x = \tilde{C}_1 \begin{pmatrix} 1 & 0 \\ 0 & 2\rho/(\sigma+\theta_\infty) \end{pmatrix}$$

with

$$\tilde{C}_0 = \begin{pmatrix} \frac{e^{\pi i(\sigma-\theta_0+\theta_x)/2} \Gamma(1-\sigma) \Gamma(-\theta_0)}{\Gamma(-\frac{\sigma+\theta_0+\theta_x}{2}) \Gamma(1-\frac{\sigma+\theta_0-\theta_x}{2})} & \frac{e^{-\pi i(\sigma+\theta_0-\theta_x)/2} \Gamma(1+\sigma) \Gamma(-\theta_0)}{\Gamma(\frac{\sigma-\theta_0-\theta_x}{2}) \Gamma(1+\frac{\sigma-\theta_0+\theta_x}{2})} \\ \frac{e^{\pi i(\sigma+\theta_0+\theta_x)/2} \Gamma(1-\sigma) \Gamma(\theta_0)}{\Gamma(-\frac{\sigma-\theta_0+\theta_x}{2}) \Gamma(1-\frac{\sigma-\theta_0-\theta_x}{2})} & \frac{e^{-\pi i(\sigma-\theta_0-\theta_x)/2} \Gamma(1+\sigma) \Gamma(\theta_0)}{\Gamma(\frac{\sigma+\theta_0-\theta_x}{2}) \Gamma(1+\frac{\sigma+\theta_0+\theta_x}{2})} \end{pmatrix},$$

$$\tilde{C}_1 = \begin{pmatrix} \frac{\Gamma(1-\sigma)\Gamma(-1-\theta_x)}{\Gamma(-\frac{\sigma-\theta_0+\theta_x}{2})\Gamma(-\frac{\sigma+\theta_0+\theta_x}{2})} & \frac{\Gamma(1+\sigma)\Gamma(-1-\theta_x)}{\Gamma(\frac{\sigma+\theta_0-\theta_x}{2})\Gamma(\frac{\sigma-\theta_0-\theta_x}{2})} \\ \frac{\Gamma(1-\sigma)\Gamma(1+\theta_x)}{\Gamma(1-\frac{\sigma-\theta_0-\theta_x}{2})\Gamma(1-\frac{\sigma+\theta_0-\theta_x}{2})} & \frac{\Gamma(1+\sigma)\Gamma(1+\theta_x)}{\Gamma(1+\frac{\sigma+\theta_0+\theta_x}{2})\Gamma(1+\frac{\sigma-\theta_0+\theta_x}{2})} \end{pmatrix}.$$

**Theorem 2.14.** Suppose that  $\theta_0, \theta_x \notin \mathbb{Z}$ . Let  $\sigma_0$  be as in Theorem 2.3. For  $\tilde{C}_0 = ((\tilde{C}_0)_{ij})$ ,  $\tilde{C}_1 = ((\tilde{C}_1)_{ij})$  and  $C_\infty$  in Theorem 2.13, set

$$(\tilde{C}_0)_{ij}(\sigma_0) := (\tilde{C}_0)_{ij}|_{\sigma=\sigma_0}, \quad (\tilde{C}_1)_{ij}(\sigma_0) := (\tilde{C}_1)_{ij}|_{\sigma=\sigma_0}, \quad C_\infty(\sigma_0) := C_\infty|_{\sigma=\sigma_0}.$$

Then the monodromy data for  $y_{\sigma_0}(\rho, x)$  of Theorem 2.3 are given by

$$M_0 = (C_0^* C_\infty(\sigma_0))^{-1} e^{\pi i \theta_0 J} C_0^* C_\infty(\sigma_0), \quad M_x = (C_x^* C_\infty(\sigma_0))^{-1} e^{\pi i \theta_x J} C_x^* C_\infty(\sigma_0)$$

with

$$C_0^* = \tilde{C}_0^* \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}, \quad C_x^* = \tilde{C}_1^* \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}.$$

Here  $\tilde{C}_0^*$  and  $\tilde{C}_1^*$  are as follows:

(i) if  $\sigma_0 = \theta_0 + \theta_x$ ,

$$\tilde{C}_0^* = \begin{pmatrix} (\tilde{C}_0)_{11}(\sigma_0) & (\tilde{C}_0^*)_{12} \\ 0 & (\tilde{C}_0)_{22}(\sigma_0) \end{pmatrix}, \quad \tilde{C}_1^* = \begin{pmatrix} (\tilde{C}_1)_{11}(\sigma_0) & (\tilde{C}_1^*)_{12} \\ 0 & (\tilde{C}_1)_{22}(\sigma_0) \end{pmatrix}$$

with

$$(\tilde{C}_0^*)_{12} = -\frac{\sigma_0 e^{-\pi i \theta_0} \Gamma(1+\sigma_0) \Gamma(-\theta_0)}{\theta_0 \Gamma(1+\theta_x)}, \quad (\tilde{C}_1^*)_{12} = -\frac{\sigma_0 \Gamma(1+\sigma_0) \Gamma(-1-\theta_x)}{\Gamma(1+\theta_0)};$$

(ii) if  $\sigma_0 = \theta_0 - \theta_x$ ,

$$\tilde{C}_0^* = \begin{pmatrix} (\tilde{C}_0)_{11}(\sigma_0) & (\tilde{C}_0)_{12}(\sigma_0)/\theta_x \\ 0 & (\tilde{C}_0)_{22}(\sigma_0) \end{pmatrix}, \quad \tilde{C}_1^* = \begin{pmatrix} 0 & (\tilde{C}_1)_{12}(\sigma_0)/\theta_x \\ (\tilde{C}_1)_{21}(\sigma_0)\theta_x & (\tilde{C}_1)_{22}(\sigma_0) \end{pmatrix};$$

(iii) if  $\sigma_0 = \theta_x - \theta_0$ ,

$$\tilde{C}_0^* = \begin{pmatrix} 0 & (\tilde{C}_0)_{12}(\sigma_0)\sigma_0/\theta_0 \\ (\tilde{C}_0)_{21}(\sigma_0)\theta_0/\sigma_0 & (\tilde{C}_0^*)_{22} \end{pmatrix}, \quad \tilde{C}_1^* = \begin{pmatrix} (\tilde{C}_1)_{11}(\sigma_0) & (\tilde{C}_1^*)_{12} \\ 0 & (\tilde{C}_1)_{22}(\sigma_0) \end{pmatrix}$$

with

$$(\tilde{C}_0^*)_{22} = \frac{e^{\pi i \theta_0} \Gamma(1+\sigma_0) \Gamma(\theta_0)}{\Gamma(1+\theta_x)}, \quad (\tilde{C}_1^*)_{12} = -\frac{\sigma_0 \Gamma(1+\sigma_0) \Gamma(-1-\theta_x)}{\Gamma(1-\theta_0)}.$$

Let  $\psi(x) = \Gamma'(x)/\Gamma(x)$ .

**Theorem 2.15.** Suppose that  $\theta_0, \theta_x \notin \mathbb{Z}$ . Then the monodromy data for  $y_{\text{ilog}}(\rho, x)$  or  $y_{\text{ilog}}^{(l)}(\rho, x)$  ( $l = 1, 2$ ) of Theorem 2.4 with  $\theta_0^2 - \theta_x^2 \neq 0$  are given by

$$M_0 = (C_0^- C_\infty)^{-1} e^{\pi i \theta_0 J} C_0^- C_\infty, \quad M_x = (C_x^- C_\infty)^{-1} e^{\pi i \theta_x J} C_x^- C_\infty,$$

those for  $y_{\text{ilog}}^\pm(\rho, x)$  with  $\theta_0^2 - \theta_x^2 = 0$  by

$$M_0 = (C_0^\pm C_\infty)^{-1} e^{\pi i \theta_0 J} C_0^\pm C_\infty, \quad M_x = (C_x^\pm C_\infty)^{-1} e^{\pi i \theta_x J} C_x^\pm C_\infty,$$

and those for  $y_{\text{ilog}}^{(1)}(\rho, x)$  with  $\theta_0 = -\theta_x \neq 0$  by

$$M_0 = (C_0^+ C_\infty)^{-1} e^{\pi i \theta_0 J} C_0^+ C_\infty, \quad M_x = (C_x^+ C_\infty)^{-1} e^{\pi i \theta_x J} C_x^+ C_\infty,$$

where

$$C_0^\pm = \tilde{C}_0^\pm \rho_0^{-\Delta}, \quad C_x^\pm = \tilde{C}_1^\pm \rho_0^{-\Delta}$$

with  $\rho_0 = \rho \exp(-2\theta_x(\theta_0^2 - \theta_x^2)^{-1})$  if  $\theta_0^2 - \theta_x^2 \neq 0$ ,  $\rho_0 = \rho$  otherwise. The matrices  $C_\infty$ ,  $\tilde{C}_0^\pm$ ,  $\tilde{C}_1^\pm$  are as follows:

(i) if  $\theta_\infty \neq 0$ , then

$$C_\infty = \begin{pmatrix} \frac{e^{-\pi i \theta_\infty/2}(\psi(1 + \theta_\infty/2) - 2\psi(1) - \pi i)}{\Gamma(1 + \theta_\infty/2)} & \frac{\psi(-\theta_\infty/2) - 2\psi(1)}{\Gamma(1 - \theta_\infty/2)} \\ \frac{e^{-\pi i \theta_\infty/2}}{\Gamma(1 + \theta_\infty/2)} & \frac{1}{\Gamma(1 - \theta_\infty/2)} \end{pmatrix},$$

and if  $\theta_\infty = 0$ , then  $C_\infty = I - \psi(1)\Delta$  for  $l = 1$ , and  $C_\infty = (1 - \pi i - \psi(1))(I + J)/2 + \Delta + \Delta_-$  for  $l = 2$ ;

(ii)

$$\tilde{C}_0^\pm = \begin{pmatrix} \frac{e^{-\pi i(\theta_0 \pm \theta_x)/2} \Gamma(-\theta_0)}{\Gamma(-(\theta_0 \mp \theta_x)/2) \Gamma(1 - (\theta_0 \pm \theta_x)/2)} & \frac{e^{-\pi i(\theta_0 \pm \theta_x)/2} \psi_{12}^0(\theta_0, \theta_x) \Gamma(-\theta_0)}{\Gamma(-(\theta_0 \mp \theta_x)/2) \Gamma(1 - (\theta_0 \pm \theta_x)/2)} \\ \frac{e^{\pi i(\theta_0 \mp \theta_x)/2} \Gamma(\theta_0)}{\Gamma(1 + (\theta_0 \mp \theta_x)/2) \Gamma((\theta_0 \pm \theta_x)/2)} & \frac{e^{\pi i(\theta_0 \mp \theta_x)/2} \psi_{22}^0(\theta_0, \theta_x) \Gamma(\theta_0)}{\Gamma(1 + (\theta_0 \mp \theta_x)/2) \Gamma((\theta_0 \pm \theta_x)/2)} \end{pmatrix}$$

with

$$\psi_{12}^0(\theta_0, \theta_x) = \psi(-(\theta_0 \mp \theta_x)/2) + \psi(1 - (\theta_0 \pm \theta_x)/2) - 2\psi(1) + \pi i,$$

$$\psi_{22}^0(\theta_0, \theta_x) = \psi(1 + (\theta_0 \mp \theta_x)/2) + \psi((\theta_0 \pm \theta_x)/2) - 2\psi(1) + \pi i,$$

and

$$\tilde{C}_1^\pm = K^\pm \times \begin{pmatrix} \frac{\Gamma(-1 \pm \theta_x)}{\Gamma(-(\theta_0 \mp \theta_x)/2) \Gamma((\theta_0 \pm \theta_x)/2)} & \frac{\psi_{12}^1(\theta_0, \theta_x) \Gamma(-1 \pm \theta_x)}{\Gamma(-(\theta_0 \mp \theta_x)/2) \Gamma((\theta_0 \pm \theta_x)/2)} \\ \frac{\Gamma(1 \mp \theta_x)}{\Gamma(1 + (\theta_0 \mp \theta_x)/2) \Gamma(1 - (\theta_0 \pm \theta_x)/2)} & \frac{\psi_{22}^1(\theta_0, \theta_x) \Gamma(1 \mp \theta_x)}{\Gamma(1 + (\theta_0 \mp \theta_x)/2) \Gamma(1 - (\theta_0 \pm \theta_x)/2)} \end{pmatrix}$$

with  $K^- = I$ ,  $K^+ = \Delta + \Delta_-$ ,

$$\psi_{12}^1(\theta_0, \theta_x) = \psi(-(\theta_0 \mp \theta_x)/2) + \psi((\theta_0 \pm \theta_x)/2) - 2\psi(1),$$

$$\psi_{22}^1(\theta_0, \theta_x) = \psi(1 + (\theta_0 \mp \theta_x)/2) + \psi(1 - (\theta_0 \pm \theta_x)/2) - 2\psi(1).$$

*Remark 2.11.* In  $\tilde{C}_0^\pm$  and  $\tilde{C}_1^\pm$  above, if  $(\theta_0 \mp \theta_x)/2 \in \mathbb{Z}$ , read  $\psi(-n)/\Gamma(-n) = (-1)^{n+1}n!$  for  $n = 0, 1, 2, \dots$ . For example, if  $(\theta_0 \mp \theta_x)/2 = 0$ , then

$$\tilde{C}_0^\pm = \begin{pmatrix} 0 & e^{-\pi i \theta_0}/\theta_0 \\ 1 & \pi i + \psi(\theta_0) - \psi(1) \end{pmatrix}, \quad \tilde{C}_1^\pm = \begin{pmatrix} 0 & 1/(1 - \theta_0) \\ 1 & \psi(1 - \theta_0) - \psi(1) \end{pmatrix}.$$



*Remark 2.12.* For  $\theta_0 \in \mathbb{Z}$  or  $\theta_x \in \mathbb{Z}$  as well, it is possible to compute the monodromy data  $M_0, M_x$  by using Propositions 9.6 through 9.8 instead of Propositions 9.3 and 9.5 in an argument of Section 10.

**Theorem 2.16.** *If  $\theta_0 \neq 0$ , the monodromy data for  $y_{\text{Taylor}}^\pm(a, x)$  of Theorem 2.5 are given by  $M_0 = Te^{\pi i \theta_0 J} T^{-1}$ ,  $M_x = Te^{-\pi i \theta_0 J} T^{-1}$ , and if  $\theta_0 = 0$ , then  $M_0 = T_0 e^{2\pi i \Delta} T_0^{-1}$ ,  $M_x = T_0 e^{-2\pi i \Delta} T_0^{-1}$ , where*

$$T = \begin{pmatrix} 1 & 1 \\ a & \theta_0 + a \end{pmatrix}, \quad T_0 = \begin{pmatrix} 1 & 1 \\ a & a - 1 \end{pmatrix}.$$

*Remark 2.13.* The transformation  $Y = DZ$  with an invertible matrix  $D$  preserves the form of (1.1) and its properties (a), (b) if and only if  $D$  is diagonal. Then (1.1) becomes

$$(2.12) \quad \frac{dZ}{d\lambda} = \left( \frac{\tilde{A}_0(x)}{\lambda} + \frac{\tilde{A}_x(x)}{\lambda - x} + \frac{J}{2} \right) Z$$

with  $\tilde{A}_0(x) = D_0^{-1} A_0(x) D_0$ ,  $\tilde{A}_x(x) = D_0^{-1} A_x(x) D_0$ ,  $D_0 = \text{diag}[r, 1/r]$ ,  $r \neq 0$ , and we have

$$y(x) = \frac{A_x(x)_{12}(A_0(x)_{11} + \theta_0/2)}{A_0(x)_{12}(A_x(x)_{11} + \theta_x/2)} = \frac{\tilde{A}_x(x)_{12}(\tilde{A}_0(x)_{11} + \theta_0/2)}{\tilde{A}_0(x)_{12}(\tilde{A}_x(x)_{11} + \theta_x/2)},$$

that is,  $(\tilde{A}_0(x), \tilde{A}_x(x)) = (D_0^{-1} A_0(x) D_0, D_0^{-1} A_x(x) D_0)$  yields the same solution as that obtained from  $(A_0(x), A_x(x))$  given in each of Theorems 2.1 through 2.5 (cf. Section 5). Hence,  $(M_0, M_x)$  for each solution above may be replaced by  $(D_0 M_0 D_0^{-1}, D_0 M_x D_0^{-1})$ , which is the monodromy for the matrix solution  $Z(\lambda, x) = (I + O(\lambda^{-1})) e^{(\lambda/2)J} \lambda^{-(\theta_\infty/2)J}$  of (2.12) (for example, cf. [2, Theorem 6.2]).

### 3. SOLUTIONS OF THE SCHLESINGER EQUATION

We give matrix solutions of (2.10), from which Theorems 2.1, 2.3 and 2.4 are obtained by using (2.11).

Suppose that  $\Lambda_0, \Lambda_x \in M_2(\mathbb{Q}_\theta[\sigma, \sigma^{-1}])$ ,  $T \in GL_2(\mathbb{Q}_\theta[\sigma, \sigma^{-1}])$  and  $\Lambda := \Lambda_0 + \Lambda_x$  have the properties:

- (P.1) the eigenvalues of  $\Lambda_\iota$  ( $\iota = 0, x$ ) are  $\pm \theta_\iota/2$ ;
- (P.2)  $(\Lambda)_{11} = (\Lambda_0 + \Lambda_x)_{11} = -(\Lambda)_{22} = -(\Lambda_0 + \Lambda_x)_{22} = -\theta_\infty/2$ ;
- (P.3)  $T^{-1} \Lambda T = (\sigma/2)J$ .

For  $\Sigma_0$  and  $\Sigma_+$  as in Theorems 2.1 and 2.3 we have

**Proposition 3.1.** (1) *System (2.10) possesses a two-parameter family of solutions  $\{(A_0(\sigma, \rho, x), A_x(\sigma, \rho, x)); (\sigma, \rho) \in \Sigma_0 \times (\mathbb{C} \setminus \{0\})\}$  given by the convergent series*

$$A_0(\sigma, \rho, x) = (\rho x^\sigma)^{\Lambda/\sigma} \left( \Lambda_0 + \sum_{n=1}^{\infty} x^n \Pi_0^n(\sigma, \rho x^\sigma) \right) (\rho x^\sigma)^{-\Lambda/\sigma},$$

$$A_x(\sigma, \rho, x) = (\rho x^\sigma)^{\Lambda/\sigma} \left( \Lambda_x + \sum_{n=1}^{\infty} x^n \Pi_x^n(\sigma, \rho x^\sigma) \right) (\rho x^\sigma)^{-\Lambda/\sigma}$$

with

$$\Pi_\iota^n(\sigma, \xi) = \sum_{m=-n}^n C_{\iota m}^n(\sigma) \xi^m, \quad C_{\iota m}^n(\sigma) \in M_2(\mathbb{Q}_\theta(\sigma))$$

$(\iota = 0, x)$ , which are holomorphic in  $(\sigma, \rho, x) \in \Omega(\Sigma_0, \varepsilon_0) \subset \Sigma_0 \times (\mathbb{C} \setminus \{0\}) \times \mathcal{R}(\mathbb{C} \setminus \{0\})$  with

$$\begin{aligned} \Omega(\Sigma_0, \varepsilon_0) &:= \bigcup_{(\sigma, \rho) \in \Sigma_0 \times (\mathbb{C} \setminus \{0\})} \{(\sigma, \rho)\} \times \Omega_{\sigma, \rho}(\varepsilon_0), \\ \Omega_{\sigma, \rho}(\varepsilon_0) &= \{x \in \mathcal{R}(\mathbb{C} \setminus \{0\}); |x(\rho x^\sigma)| < \varepsilon_0, |x(\rho x^\sigma)^{-1}| < \varepsilon_0\}, \end{aligned}$$

$\varepsilon_0$  being a sufficiently small number depending only on  $\Sigma_0$  and  $(\theta_0, \theta_x, \theta_\infty)$ .

(2) If  $(T^{-1}\Lambda_0 T)_{21}$  vanishes at  $\sigma = \sigma_0 \in \Sigma_+$ , then (2.10) admits a one-parameter family of solutions  $\{(A_0(\sigma_0, \rho, x), A_x(\sigma_0, \rho, x)); \rho \in \mathbb{C}\}$  given by the representations above restricted to  $\sigma = \sigma_0$  whose inner sums satisfy

$$\xi^{\Lambda/\sigma_0} \Pi_\iota^n(\sigma_0, \xi) \xi^{-\Lambda/\sigma_0} = \sum_{m=0}^{n+1} \tilde{C}_{\iota m}^n(\sigma_0) \xi^m, \quad \xi^{\Lambda/\sigma_0} \Lambda_\iota \xi^{-\Lambda/\sigma_0} = \tilde{C}_{\iota 0}^0(\sigma_0) + \tilde{C}_{\iota 1}^0(\sigma_0) \xi$$

with  $\tilde{C}_{\iota m}^n(\sigma_0) \in M_2(\mathbb{Q}_\theta(\sigma_0))$  ( $\iota = 0, x$ ) for  $n \geq 0$ . Each entry of the solution is holomorphic in  $(\rho, x) \in \Omega(\varepsilon_0) \subset \mathbb{C} \times \mathcal{R}(\mathbb{C} \setminus \{0\})$ , where

$$\begin{aligned} \Omega(\varepsilon_0) &:= \bigcup_{\rho \in \mathbb{C}} \{\rho\} \times \Omega_\rho(\varepsilon_0), \\ \Omega_\rho(\varepsilon_0) &:= \{x \in \mathcal{R}(\mathbb{C} \setminus \{0\}); |x| < \varepsilon_0, |x(\rho x^{\sigma_0})| < \varepsilon_0\}, \end{aligned}$$

$\varepsilon_0$  being a sufficiently small number depending only on  $\sigma_0$  and  $(\theta_0, \theta_x, \theta_\infty)$ .

Set  $\mathbb{Q}_{\tilde{\theta}} := \mathbb{Q}_\theta[\theta_\infty^{-1}] = \mathbb{Q}[\theta_0, \theta_x, \theta_\infty, \theta_\infty^{-1}]$  if  $\theta_\infty \neq 0$ , and  $= \mathbb{Q}[\theta_0, \theta_x]$  if  $\theta_\infty = 0$ . Let  $\Lambda_0, \Lambda_x \in M_2(\mathbb{Q}_{\tilde{\theta}})$ ,  $T \in GL_2(\mathbb{Q}_{\tilde{\theta}})$  and  $\Lambda := \Lambda_0 + \Lambda_x$  have the properties (P.1), (P.2) and (P.3')  $T^{-1}\Lambda T = \Delta$ .

Then we have

**Proposition 3.2.** *System (2.10) possesses a one-parameter family of solutions of logarithmic type  $\{(A_0(\rho, x), A_x(\rho, x)); \rho \in \mathcal{R}(\mathbb{C} \setminus \{0\})\}$  given by the convergent series*

$$\begin{aligned} A_0(\rho, x) &= (\rho x)^\Lambda \left( \Lambda_0 + \sum_{n=1}^{\infty} x^n \Pi_0^{*n}(\log(\rho x)) \right) (\rho x)^{-\Lambda}, \\ A_x(\rho, x) &= (\rho x)^\Lambda \left( \Lambda_x + \sum_{n=1}^{\infty} x^n \Pi_x^{*n}(\log(\rho x)) \right) (\rho x)^{-\Lambda} \end{aligned}$$

with

$$\Pi_\iota^{*n}(\xi) = \sum_{m=0}^{2n} C_{\iota m}^{*n} \xi^m, \quad C_{\iota m}^{*n} \in M_2(\mathbb{Q}_{\tilde{\theta}})$$

( $\iota = 0, x$ ), which are holomorphic in  $(\rho, x) \in \Omega^*(\varepsilon_0, \Theta_0) \subset \mathcal{R}(\mathbb{C} \setminus \{0\})^2$ , where  $\Omega^*(\varepsilon_0, \Theta_0)$  is the domain as in Theorem 2.4. Furthermore, if  $(T^{-1}\Lambda_0 T)_{21} = 0$ , then

$$e^{\Lambda\xi}\Pi_\iota^{*n}(\xi)e^{-\Lambda\xi} = \sum_{m=0}^{n+1} \tilde{C}_{\iota m}^{*n}\xi^m, \quad e^{\Lambda\xi}\Lambda_\iota e^{-\Lambda\xi} = \tilde{C}_{\iota 0}^{*0} + \tilde{C}_{\iota 1}^{*0}\xi$$

with  $\tilde{C}_{\iota m}^{*n} \in M_2(\mathbb{Q}_{\hat{\theta}})$  ( $\iota = 0, x$ ) for  $n \geq 0$ .

System (2.10) corresponds to the Schlesinger equation associated with the sixth Painlevé equation studied in [19]. These propositions are proved by the same arguments as in the proofs of [19, Theorems 2.1 and 2.2] given in [19, §5]. We describe the outline of the proofs of them. By the change of variables

$$(3.1) \quad x = \kappa t, \quad A_0 = t^\Lambda(\Lambda_0 + U_0)t^{-\Lambda}, \quad A_0 + A_x = \Lambda + U_\infty,$$

where  $\Lambda = \Lambda_0 + \Lambda_x$  and  $\kappa \neq 0$ , equation (2.10) is taken to

$$(3.2) \quad \begin{aligned} t \frac{dU_0}{dt} &= [t^{-\Lambda}U_\infty t^\Lambda, \Lambda_0 + U_0], \\ t \frac{dU_\infty}{dt} &= \kappa t[J/2, t^\Lambda \Lambda_x t^{-\Lambda} - t^\Lambda U_0 t^{-\Lambda} + U_\infty], \end{aligned}$$

since  $A_x = \Lambda + U_\infty - t^\Lambda(\Lambda_0 + U_0)t^{-\Lambda} = t^\Lambda(\Lambda_x - U_0 + t^{-\Lambda}U_\infty t^\Lambda)t^{-\Lambda}$ . The form of system (3.2) is similar to that of [19, (5.2)].

To show Proposition 3.2 for each fixed  $(\theta_0, \theta_x, \theta_\infty)$ , we use the ring  $\widehat{\mathfrak{L}}$  of formal series of the form

$$\Phi = \Phi(\kappa, t) = \sum_{n=1}^{\infty} \sum_{m=0}^{2n} C_m^n (\kappa t)^n \log^m t, \quad C_m^n \in M_2(\mathbb{Q}_{\hat{\theta}}),$$

and the subring

$$\mathfrak{L}(D) := \{\Phi \in \widehat{\mathfrak{L}}; \|\Phi\| < \infty \text{ for } (\kappa, t) \in D\},$$

which are defined in [19, §4.1]. Here, for  $\Phi \in \widehat{\mathfrak{L}}$  as above,  $\|\Phi\|$  is the norm defined by

$$\|\Phi\| := \sum_{n=1}^{\infty} \sum_{m=0}^{2n} \|C_m^n\| |\kappa t|^n |t|^{-m/4}$$

with the standard norm of the matrix  $\|C_m^n\| = \max_{i=1,2} \{|(C_m^n)_{i1}| + |(C_m^n)_{i2}|\}$ , and  $D$  is a subdomain of  $(\mathbb{C} \setminus \{0\}) \times \mathcal{R}(\mathbb{C} \setminus \{0\})$  such that  $|\log t| \leq |t|^{-1/4}$  for every  $(\kappa, t) \in D$ . Then the holomorphic nature of  $\Phi(\kappa, t) \in \mathfrak{L}(D)$  in  $D$  is guaranteed by [19, Proposition 4.1]. For  $(m, n) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}$  and  $C \in M_2(\mathbb{Q}_{\hat{\theta}})$  let

$$\mathcal{I}[C(\kappa t)^n \log^m t] := C \frac{(\kappa t)^n}{n} \left( \log^m t + \cdots + \frac{(-1)^j m!}{n^j (m-j)!} \log^{m-j} t + \cdots + \frac{(-1)^m m!}{n^m} \right).$$

Then  $\mathcal{I}$  induces a linear operator  $\mathcal{I} : \widehat{\mathfrak{L}} \rightarrow \widehat{\mathfrak{L}}$  assigning the formal primitive function of  $t^{-1}\Phi$  to each  $\Phi \in \widehat{\mathfrak{L}}$ , which satisfies  $\mathcal{I}[\Phi] \in \mathfrak{L}(D)$ ,  $t(d/dt)\mathcal{I}[\Phi] = \Phi$  and  $\|\mathcal{I}[\Phi]\| \leq 2\|\Phi\|$  for  $\Phi \in \mathfrak{L}(D)$ , provided that, for  $(\kappa, t) \in D$ ,  $|t|$  is sufficiently small. To construct a solution of (3.2) we define the sequence  $\{(U_0^{(\nu)}, U_\infty^{(\nu)}) \in (\widehat{\mathfrak{L}})^2; \nu \geq 0\}$  by

$$U_0^{(0)} = U_0^{(0)} \equiv 0,$$

$$(3.3) \quad \begin{aligned} U_\infty^{(\nu+1)} &= \mathcal{I}[\kappa t[J/2, t^\Lambda \Lambda_x t^{-\Lambda} - t^\Lambda U_0^{(\nu)} t^{-\Lambda} + U_\infty^{(\nu)}]], \\ U_0^{(\nu+1)} &= \mathcal{I}[t^{-\Lambda} U_\infty^{(\nu+1)} t^\Lambda, \Lambda_0 + U_0^{(\nu)}] \end{aligned}$$

with  $\Lambda_0$ ,  $\Lambda_x$  and  $\Lambda$  satisfying (P.1), (P.2), (P.3'). This converges to a formal series solution of (3.2). Choosing  $D$  suitably, we may show that  $(U_0^\infty, U_\infty^\infty) = \lim_{\nu \rightarrow \infty} (U_0^{(\nu)}, U_\infty^{(\nu)}) \in \mathfrak{L}(D)^2$  solves (3.2) and that  $t^{-\Lambda} U_\infty^\infty t^\Lambda \in \mathfrak{L}(D)$ . Setting  $\kappa t = x$  and  $t = \rho x$  in

$$A_0 = t^\Lambda (\Lambda_0 + U_0^\infty) t^{-\Lambda}, \quad A_x = t^\Lambda (\Lambda_x - U_0^\infty + t^{-\Lambda} U_\infty^\infty t^\Lambda) t^{-\Lambda},$$

we obtain the family of solutions  $\{(A_0(\sigma, \rho, x), A_x(\sigma, \rho, x))\}$  as in Proposition 3.2. If  $(T^{-1} \Lambda_0 T)_{21} = 0$ , then  $\kappa t \cdot t^\Lambda U_0^\infty t^{-\Lambda}$ ,  $U_\infty^\infty \in \widehat{\mathfrak{L}}^*$ , where  $\widehat{\mathfrak{L}}^*$  is the subring of  $\widehat{\mathfrak{L}}$  consisting of formal series of the form

$$\Phi = \sum_{n=1}^{\infty} \sum_{m=0}^n C_m^n (\kappa t)^n \log^m t, \quad C_m^n \in M_2(\mathbb{Q}_{\hat{\theta}}).$$

From this fact the remaining part of Proposition 3.2 follows (cf. [19, §5.2]).

To show Proposition 3.1 for each fixed  $(\theta_0, \theta_x, \theta_\infty)$ , consider the ring  $\widehat{\mathfrak{S}}$  of formal series of the form

$$\Phi = \Phi(\sigma, \kappa, t) = \sum_{n=1}^{\infty} \sum_{m=-n}^n C_m^n (\kappa t)^n t^{\sigma m}, \quad C_m^n = C_m^n(\sigma) \in M_2(\mathbb{Q}_\theta(\sigma)),$$

and the subring

$$\mathfrak{S}(D(\Sigma_0)) := \{\Phi \in \widehat{\mathfrak{S}}; \|\Phi\| < \infty \text{ for } (\sigma, \kappa, t) \in D(\Sigma_0)\}$$

as in [19, §4.2]. Here, for  $\Phi \in \widehat{\mathfrak{S}}$  as above,

$$\|\Phi\| := \sum_{n=1}^{\infty} \sum_{m=-n}^n \|C_m^n\| |\kappa t|^n |t^\sigma|^m,$$

and  $D(\Sigma_0)$  is a subdomain of  $\Sigma_0 \times (\mathbb{C} \setminus \{0\}) \times \mathcal{R}(\mathbb{C} \setminus \{0\})$ . For  $(m, n) \in \mathbb{Z} \times \mathbb{N}$  and  $C \in M_2(\mathbb{Q}_\theta(\sigma))$ , let  $\mathcal{I}$  be such that

$$\mathcal{I}[C(\kappa t)^n t^{\sigma m}] := \frac{1}{n + \sigma m} C(\kappa t)^n t^{\sigma m}.$$

This induces a linear operator  $\mathcal{I} : \widehat{\mathfrak{S}} \rightarrow \widehat{\mathfrak{S}}$  satisfying  $\mathcal{I}[\Phi] \in \mathfrak{S}(D(\Sigma_0))$ ,  $t(d/dt)\mathcal{I}[\Phi] = \Phi$  and  $\|\mathcal{I}[\Phi]\| \leq L_0 \|\Phi\|$  for some  $L_0 > 0$  if  $\Phi \in \mathfrak{S}(D(\Sigma_0))$ . Then we define  $\{(U_0^{(\nu)}, U_\infty^{(\nu)}) \in (\widehat{\mathfrak{S}})^2; \nu \geq 0\}$  by (3.3) with  $\Lambda_0$ ,  $\Lambda_x$  and  $\Lambda$  satisfying (P.1), (P.2), (P.3). Choosing  $D(\Sigma_0)$  suitably, we may show that  $(U_0^\infty, U_\infty^\infty) = \lim_{\nu \rightarrow \infty} (U_0^{(\nu)}, U_\infty^{(\nu)}) \in \mathfrak{S}(D(\Sigma_0))^2$  solves (3.2) and that  $t^{-\Lambda} U_\infty^\infty t^\Lambda \in \mathfrak{S}(D(\Sigma_0))$ . Setting  $\kappa t = x$  and  $t^\sigma = \rho x^\sigma$  we obtain Proposition 3.1, (1). The second assertion is proved by considering, for  $\sigma_0 \in \Sigma_+$ , the ring  $\widehat{\mathfrak{S}}^+(\sigma_0)$  of the formal series

$$\Phi(\sigma_0, \kappa, t) = \sum_{n=1}^{\infty} \sum_{m=0}^n C_m^n (\kappa t)^n t^{\sigma_0 m}, \quad C_m^n \in M_2(\mathbb{Q}_{\theta}(\sigma_0)),$$

and by using the sequence  $\{(Z_0^{(\nu)}, U_\infty^{(\nu)})|_{\sigma=\sigma_0}; \nu \geq 0\}$  with  $Z_0^{(\nu)} := t^\Lambda U_0^{(\nu)} t^{-\Lambda}$  such that  $\kappa t Z_0^{(\nu)}$ ,  $U_\infty^{(\nu)} \in \widehat{\mathfrak{S}}^+(\sigma_0)$  if  $(T^{-1} \Lambda_0 T)_{21}|_{\sigma=\sigma_0} = 0$ . In a suitable domain, we get

$(U_\infty^\infty, \kappa t Z_0^\infty) = \lim_{\nu \rightarrow \infty} (U_\infty^{(\nu)}, \kappa t Z_0^{(\nu)})$ , from which the desired solution of (2.10) follows (cf. [19, §5.3]).

**Proposition 3.3.** *The solution  $(A_0(x), A_x(x))$  of (2.10) given by Proposition 3.1 or 3.2 satisfies the conditions (a) and (b) on (1.1), and the corresponding system (1.1) has the isomonodromy property.*

*Proof.* Observing that  $(d/dx)(A_0(x) + A_x(x)) = [J, A_x(x)]/2$ , in which the diagonal part on the right-hand side vanishes identically, we deduce (b) from the fact that  $A_0(x) + A_x(x) \rightarrow \Lambda$  as  $x \rightarrow 0$  along a suitable curve. The property (a) may be verified by the same argument as in the proof of [19, Proposition 3.1].  $\square$

#### 4. LEMMAS ON MATRICES

For  $\sigma \neq 0$ , we have the following (cf. [7, Lemma A.2]):

**Lemma 4.1.** *The matrices*

$$\begin{aligned} \Lambda_0 &= T \begin{pmatrix} (\Lambda'_0)_{11} & 1 \\ (\Lambda'_0)_{21} & -(\Lambda'_0)_{11} \end{pmatrix} T^{-1}, \quad \Lambda_x = T \begin{pmatrix} (\Lambda'_x)_{11} & -1 \\ (\Lambda'_x)_{21} & -(\Lambda'_x)_{11} \end{pmatrix} T^{-1}, \\ (\Lambda'_0)_{11} &= \frac{\sigma}{2} - (\Lambda'_x)_{11} = \frac{1}{4\sigma}(\sigma^2 + \theta_0^2 - \theta_x^2), \\ -(\Lambda'_0)_{21} &= (\Lambda'_x)_{21} = \frac{1}{16\sigma^2}((\theta_0 + \theta_x)^2 - \sigma^2)((\theta_0 - \theta_x)^2 - \sigma^2), \\ T &= \begin{pmatrix} (\sigma - \theta_\infty)/2 & -1 \\ (\sigma + \theta_\infty)/2 & 1 \end{pmatrix}, \quad \Lambda = \Lambda_0 + \Lambda_x = \begin{pmatrix} -\theta_\infty/2 & (\sigma - \theta_\infty)/2 \\ (\sigma + \theta_\infty)/2 & \theta_\infty/2 \end{pmatrix} \end{aligned}$$

have the properties (P.1), (P.2) and (P.3).

Using [8, Proposition 2.1, Jordan case], we have

**Lemma 4.2.** (1) *If  $\theta_\infty \neq 0$ , then the matrices*

$$\begin{aligned} \Lambda_0 &= T \begin{pmatrix} \mp \theta_x/2 & 1 \\ (\theta_0^2 - \theta_x^2)/4 & \pm \theta_x/2 \end{pmatrix} T^{-1}, \quad \Lambda_x = T \begin{pmatrix} \pm \theta_x/2 & 0 \\ (\theta_x^2 - \theta_0^2)/4 & \mp \theta_x/2 \end{pmatrix} T^{-1}, \\ T &= \begin{pmatrix} -\theta_\infty/2 & 1 \\ \theta_\infty/2 & 0 \end{pmatrix}, \quad \Lambda = \Lambda_0 + \Lambda_x = \frac{\theta_\infty}{2} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

have the properties (P.1), (P.2) and (P.3').

(2) *If  $\theta_\infty = 0$ , then the matrices  $\Lambda_0, \Lambda_x$  given as above with*

$$T = I \quad \left( \text{respectively, } T = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \right)$$

and  $\Lambda = \Delta$  (respectively,  $\Lambda = \Delta_-$ ) have the properties (P.1), (P.2) and (P.3').

**Lemma 4.3.** *Suppose that  $\theta_\infty = 0$ .*

(1) *If  $\theta_0 = \pm\theta_x \neq 0$ , then*

$$\Lambda_0 = -\Lambda_x = T(\theta_0/2)JT^{-1} = \begin{pmatrix} \theta_0/2 + a & -1 \\ a(\theta_0 + a) & -\theta_0/2 - a \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ a & \theta_0 + a \end{pmatrix}$$

*have the properties (P.1) and (P.2).*

(2) *If  $\theta_0 = \theta_x = 0$ , then*

$$\Lambda_0 = -\Lambda_x = T\Delta T^{-1} = \begin{pmatrix} a & -1 \\ a^2 & -a \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ a & a - 1 \end{pmatrix}$$

*have the properties (P.1) and (P.2).*

## 5. PROOFS OF THEOREMS 2.1, AND 2.3 THROUGH 2.5

**5.1. Proof of Theorem 2.1.** By Proposition 3.1 with  $(\Lambda_0, \Lambda_x, T, \Lambda)$  in Lemma 4.1 we have, for  $\sigma \in \Sigma_0$ ,  $\rho \in \mathbb{C} \setminus \{0\}$ ,

$$\begin{aligned} A_0(\sigma, \rho, x) &= T(\rho x^\sigma)^{J/2} \left( T^{-1} \Lambda_0 T + \sum_{n=1}^{\infty} x^n T^{-1} \Pi_0^n(\sigma, \rho x^\sigma) T \right) (\rho x^\sigma)^{-J/2} T^{-1} \\ &= T \left( (\rho x^\sigma)^{J/2} T^{-1} \Lambda_0 T (\rho x^\sigma)^{-J/2} + \sum_{n=1}^{\infty} x^n \sum_{j=-n-1}^{n+1} \tilde{C}_{0j}^n(\sigma) (\rho x^\sigma)^j \right) T^{-1} \end{aligned}$$

with  $\tilde{C}_{0j}^n(\sigma) \in M_2(\mathbb{Q}_\theta(\sigma))$ , and hence

$$\begin{aligned} (A_0)_{11} &= -\sigma^{-1} \left( \theta_\infty (\Lambda'_0)_{11} + \frac{1}{4} (\sigma^2 - \theta_\infty^2) \rho x^\sigma + (\Lambda'_0)_{21} (\rho x^\sigma)^{-1} \right) + (\cdots), \\ (A_0)_{12} &= -\sigma^{-1} \left( -(\sigma - \theta_\infty) (\Lambda'_0)_{11} - \frac{1}{4} (\sigma - \theta_\infty)^2 \rho x^\sigma + (\Lambda'_0)_{21} (\rho x^\sigma)^{-1} \right) + (\cdots), \end{aligned}$$

where  $(\cdots)$  denotes a series of the form  $\sum_{n=1}^{\infty} x^n \sum_{j=-n-1}^{n+1} c_{nj}(\sigma) (\rho x^\sigma)^j$ ,  $c_{nj}(\sigma) \in \mathbb{Q}_\theta(\sigma)$ . Similarly,

$$\begin{aligned} (A_x)_{11} &= -\sigma^{-1} \left( \theta_\infty (\Lambda'_x)_{11} - \frac{1}{4} (\sigma^2 - \theta_\infty^2) \rho x^\sigma + (\Lambda'_x)_{21} (\rho x^\sigma)^{-1} \right) + (\cdots), \\ (A_x)_{12} &= -\sigma^{-1} \left( -(\sigma - \theta_\infty) (\Lambda'_x)_{11} + \frac{1}{4} (\sigma - \theta_\infty)^2 \rho x^\sigma + (\Lambda'_x)_{21} (\rho x^\sigma)^{-1} \right) + (\cdots). \end{aligned}$$

By Proposition 3.3 and (2.11),  $y = Y_{11}Y_{12}$  with

$$(5.1) \quad Y_{11} = \frac{(A_0)_{11} + \theta_0/2}{(A_x)_{11} + \theta_x/2}, \quad Y_{12} = \frac{(A_x)_{12}}{(A_0)_{12}}$$

solves (V). Then we may write  $Y_{ij} = (Y_{ij})_{\text{num}} / (Y_{ij})_{\text{den}}$  ( $(i, j) = (1, 1), (1, 2)$ ) with

$$\begin{aligned} (5.2) \quad (Y_{11})_{\text{num}} &= 4\theta_0\sigma^2 - 2\theta_\infty(\sigma^2 + \theta_0^2 - \theta_x^2) - 2\sigma(\sigma^2 - \theta_\infty^2)\rho x^\sigma \\ &\quad + \frac{1}{2\sigma}((\theta_0 + \theta_x)^2 - \sigma^2)((\theta_0 - \theta_x)^2 - \sigma^2)(\rho x^\sigma)^{-1} + \sigma^2(\cdots), \\ (Y_{11})_{\text{den}} &= 4\theta_x\sigma^2 - 2\theta_\infty(\sigma^2 + \theta_x^2 - \theta_0^2) + 2\sigma(\sigma^2 - \theta_\infty^2)\rho x^\sigma \\ &\quad - \frac{1}{2\sigma}((\theta_0 + \theta_x)^2 - \sigma^2)((\theta_0 - \theta_x)^2 - \sigma^2)(\rho x^\sigma)^{-1} + \sigma^2(\cdots), \end{aligned}$$

and

$$\begin{aligned}
(Y_{12})_{\text{num}} &= 2(\sigma - \theta_\infty)(\sigma^2 + \theta_x^2 - \theta_0^2) - 2\sigma(\sigma - \theta_\infty)^2 \rho x^\sigma \\
&\quad - \frac{1}{2\sigma}((\theta_0 + \theta_x)^2 - \sigma^2)((\theta_0 - \theta_x)^2 - \sigma^2)(\rho x^\sigma)^{-1} + \sigma^2(\cdots), \\
(Y_{12})_{\text{den}} &= 2(\sigma - \theta_\infty)(\sigma^2 + \theta_0^2 - \theta_x^2) + 2\sigma(\sigma - \theta_\infty)^2 \rho x^\sigma \\
&\quad + \frac{1}{2\sigma}((\theta_0 + \theta_x)^2 - \sigma^2)((\theta_0 - \theta_x)^2 - \sigma^2)(\rho x^\sigma)^{-1} + \sigma^2(\cdots).
\end{aligned} \tag{5.3}$$

Aiming at a solution of the form as in Theorem 2.1 we replace  $\rho$  by

$$\frac{(\theta_0 - \theta_x + \sigma)(\theta_0 + \theta_x - \sigma)}{2\sigma(\sigma + \theta_\infty)} \rho. \tag{5.4}$$

Then  $(Y_{ij})_{\text{num}}$  and  $(Y_{ij})_{\text{den}}$  become

$$\begin{aligned}
(Y_{11})_{\text{num}} &= 4\theta_0\sigma^2 - 2\theta_\infty(\sigma^2 + \theta_0^2 - \theta_x^2) - (\sigma - \theta_\infty)(\theta_0^2 - (\sigma - \theta_x)^2) \rho x^\sigma \\
&\quad + (\sigma + \theta_\infty)(\theta_0^2 - (\sigma + \theta_x)^2)(\rho x^\sigma)^{-1} + (\cdots), \\
(Y_{11})_{\text{den}} &= 4\theta_x\sigma^2 - 2\theta_\infty(\sigma^2 - \theta_0^2 + \theta_x^2) + (\sigma - \theta_\infty)(\theta_0^2 - (\sigma - \theta_x)^2) \rho x^\sigma \\
&\quad - (\sigma + \theta_\infty)(\theta_0^2 - (\sigma + \theta_x)^2)(\rho x^\sigma)^{-1} + (\cdots),
\end{aligned} \tag{5.5}$$

and  $(Y_{12})_{\text{num}} = (\sigma + \theta_\infty)^{-1}(Y_{12})_{\text{num}}^*$ ,  $(Y_{12})_{\text{den}} = (\sigma + \theta_\infty)^{-1}(Y_{12})_{\text{den}}^*$  with

$$\begin{aligned}
(Y_{12})_{\text{num}}^* &= 2(\sigma^2 - \theta_\infty^2)(\sigma^2 - \theta_0^2 + \theta_x^2) - (\sigma - \theta_\infty)^2(\theta_0^2 - (\sigma - \theta_x)^2) \rho x^\sigma \\
&\quad - (\sigma + \theta_\infty)^2(\theta_0^2 - (\sigma + \theta_x)^2)(\rho x^\sigma)^{-1} + (\cdots), \\
(Y_{12})_{\text{den}}^* &= 2(\sigma^2 - \theta_\infty^2)(\sigma^2 + \theta_0^2 - \theta_x^2) + (\sigma - \theta_\infty)^2(\theta_0^2 - (\sigma - \theta_x)^2) \rho x^\sigma \\
&\quad + (\sigma + \theta_\infty)^2(\theta_0^2 - (\sigma + \theta_x)^2)(\rho x^\sigma)^{-1} + (\cdots),
\end{aligned} \tag{5.6}$$

which are holomorphic in a domain on  $\mathcal{R}(\mathbb{C} \setminus \{0\})$  where  $|x(\rho x^\sigma)|$  and  $|x(\rho x^\sigma)^{-1}|$  are sufficiently small. A general solution  $y(\sigma, \rho, x)$  meromorphic in such a domain is represented in terms of (5.5) and (5.6). If  $\rho x^\sigma$  is also sufficiently small, then, under the supposition of Theorem 2.1, we may write, say  $(Y_{11})_{\text{num}}$ , in the form

$$\begin{aligned}
(Y_{11})_{\text{num}} &= (\sigma + \theta_\infty)(\theta_0^2 - (\sigma + \theta_x)^2)(\rho x^\sigma)^{-1} \left( 1 + \frac{4\theta_0\sigma^2 - 2\theta_\infty(\sigma^2 + \theta_0^2 - \theta_x^2)}{(\sigma + \theta_\infty)(\theta_0^2 - (\sigma + \theta_x)^2)} \rho x^\sigma \right. \\
&\quad \left. + c_2^{(11)}(\sigma)(\rho x^\sigma)^2 + \sum_{n=1}^{\infty} x^n \sum_{j=-n-1}^{n+1} c_{nj}^{(11)}(\sigma)(\rho x^\sigma)^{j+1} \right)
\end{aligned}$$

with  $c_2^{(11)}(\sigma), c_{nj}^{(11)}(\sigma) \in \mathbb{Q}_\theta(\sigma)$ . From this and analogous expressions of  $(Y_{11})_{\text{den}}, (Y_{12})_{\text{num}}^*$  and  $(Y_{12})_{\text{den}}^*$ , we derive  $y_+(\sigma, \rho, x)$  convergent in  $\Omega^+(\Sigma_0, \varepsilon_0)$ . If  $(\rho x^\sigma)^{-1}$  is sufficiently small, writing

$$\begin{aligned}
(Y_{11})_{\text{num}} &= -(\sigma - \theta_\infty)(\theta_0^2 - (\sigma - \theta_x)^2) \rho x^\sigma \left( 1 - \frac{4\theta_0\sigma^2 - 2\theta_\infty(\sigma^2 + \theta_0^2 - \theta_x^2)}{(\sigma - \theta_\infty)(\theta_0^2 - (\sigma - \theta_x)^2)} (\rho x^\sigma)^{-1} \right. \\
&\quad \left. + c_2'^{(11)}(\sigma)(\rho x^\sigma)^{-2} + \sum_{n=1}^{\infty} x^n \sum_{j=-n-1}^{n+1} c_{nj}'^{(11)}(\sigma)(\rho x^\sigma)^{-j-1} \right)
\end{aligned}$$

with  $c_2'^{(11)}(\sigma)$ ,  $c_{nj}'^{(11)}(\sigma) \in \mathbb{Q}_\theta(\sigma)$  and so on, we have  $y_-(\sigma, \rho, x)$ . Thus we obtain Theorem 2.1.

**5.2. Proof of Theorem 2.3.** If  $\sigma_0^2 = (\theta_0 \pm \theta_x)^2$ , then  $2\theta_0\sigma_0^2 - \theta_\infty(\sigma_0^2 + \theta_0^2 - \theta_x^2) = 2\theta_0(\theta_0 \pm \theta_x)(\theta_0 \pm \theta_x - \theta_\infty)$ . Putting  $\sigma = \sigma_0$  in  $Y_{11}$  and  $Y_{12}$  with (5.2) and (5.3), we have

$$(5.7) \quad \begin{aligned} Y_{11}|_{\sigma=\sigma_0} &= \frac{2\theta_0(\theta_0 \pm \theta_x)(\theta_0 \pm \theta_x - \theta_\infty) - \sigma_0(\sigma_0^2 - \theta_\infty^2)\rho x^{\sigma_0} + \sigma_0^2(\cdots)}{2\theta_x(\theta_x \pm \theta_0)(\theta_x \pm \theta_0 - \theta_\infty) + \sigma_0(\sigma_0^2 - \theta_\infty^2)\rho x^{\sigma_0} + \sigma_0^2(\cdots)}, \\ Y_{12}|_{\sigma=\sigma_0} &= \frac{2\theta_x(\theta_x \pm \theta_0) - \sigma_0(\sigma_0 - \theta_\infty)\rho x^{\sigma_0} + \sigma_0^2(\sigma_0 - \theta_\infty)^{-1}(\cdots)}{2\theta_0(\theta_0 \pm \theta_x) + \sigma_0(\sigma_0 - \theta_\infty)\rho x^{\sigma_0} + \sigma_0^2(\sigma_0 - \theta_\infty)^{-1}(\cdots)} \end{aligned}$$

with  $(\cdots)$  denoting a series of the form  $\sum_{n=1}^{\infty} x^n \sum_{j=0}^{n+1} c_{nj}(\sigma_0)(\rho x^{\sigma_0})^j$ . Suppose that  $\sigma_0 = \theta_0 + \theta_x \neq \theta_\infty$  and  $\theta_0\theta_x \neq 0$ . If  $\rho x^{\sigma_0}$  is sufficiently small,

$$\begin{aligned} Y_{11}Y_{12}|_{\sigma=\sigma_0} &= \frac{1 - (\sigma_0 + \theta_\infty)\rho x^{\sigma_0}/(2\theta_0) + \cdots}{1 + (\sigma_0 + \theta_\infty)\rho x^{\sigma_0}/(2\theta_x) + \cdots} \cdot \frac{1 - (\sigma_0 - \theta_\infty)\rho x^{\sigma_0}/(2\theta_x) + \cdots}{1 + (\sigma_0 - \theta_\infty)\rho x^{\sigma_0}/(2\theta_0) + \cdots} \\ &= 1 - \frac{\sigma_0^2}{\theta_0\theta_x}\rho x^{\sigma_0} + \sum_{j \geq 2} c_j^0(\sigma_0)(\rho x^{\sigma_0})^j + \sum_{n=1}^{\infty} x^n \sum_{j \geq 0} c_{jn}^0(\sigma_0)(\rho x^{\sigma_0})^j \end{aligned}$$

with  $c_j^0(\sigma_0)$ ,  $c_{jn}^0(\sigma_0) \in \mathbb{Q}_\theta[\theta_0^{-1}, \theta_x^{-1}](\sigma_0)$ , and if  $(\rho x^{\sigma_0})^{-1}$  and  $x(\rho x^{\sigma_0})$  are sufficiently small,

$$\begin{aligned} &= \frac{1 - 2\theta_0(\rho x^{\sigma_0})^{-1}/(\sigma_0 + \theta_\infty) + \cdots}{1 + 2\theta_x(\rho x^{\sigma_0})^{-1}/(\sigma_0 + \theta_\infty) + \cdots} \cdot \frac{1 - 2\theta_x(\rho x^{\sigma_0})^{-1}/(\sigma_0 - \theta_\infty) + \cdots}{1 + 2\theta_0(\rho x^{\sigma_0})^{-1}/(\sigma_0 - \theta_\infty) + \cdots} \\ &= 1 - \frac{4\sigma_0^2}{\sigma_0^2 - \theta_\infty^2}(\rho x^{\sigma_0})^{-1} + \sum_{j \geq 2} \tilde{c}_j^0(\sigma_0)(\rho x^{\sigma_0})^{-j} + \sum_{n=1}^{\infty} x^n \sum_{j \geq 0} \tilde{c}_{jn}^0(\sigma_0)(\rho x^{\sigma_0})^{n-j} \end{aligned}$$

with  $\tilde{c}_j^0(\sigma_0)$ ,  $\tilde{c}_{jn}^0(\sigma_0) \in \mathbb{Q}_\theta(\sigma_0)$ . The other cases are similarly treated.

**5.3. Proof of Theorem 2.4.** Suppose that  $\theta_\infty \neq 0$ . By Proposition 3.2 with  $(\Lambda_0, \Lambda_x, T, \Lambda)$  in Lemma 4.2, (1), we have

$$A_0(\rho, x) = T \left( (\rho x)^\Delta T^{-1} \Lambda_0 T (\rho x)^{-\Delta} + \sum_{n=1}^{\infty} x^n (\rho x)^\Delta T^{-1} \Pi_0^{*n}(\log(\rho x)) T (\rho x)^{-\Delta} \right) T^{-1}.$$

Hence

$$\begin{aligned} (A_0)_{11} &= \pm \frac{\theta_x}{2} - \frac{\theta_\infty}{2} - \frac{1}{4}(\theta_0^2 - \theta_x^2 \pm 2\theta_x\theta_\infty) \log(\rho x) + \frac{1}{8}(\theta_0^2 - \theta_x^2)\theta_\infty \log^2(\rho x) + (\cdots), \\ (A_0)_{12} &= \pm \theta_x - \frac{\theta_\infty}{2} + \frac{\theta_0^2 - \theta_x^2}{2\theta_\infty} - \frac{1}{2}(\theta_0^2 - \theta_x^2 \pm \theta_x\theta_\infty) \log(\rho x) \\ &\quad + \frac{1}{8}(\theta_0^2 - \theta_x^2)\theta_\infty \log^2(\rho x) + (\cdots), \end{aligned}$$

and

$$(A_x)_{11} = -(A_0)_{11} - \frac{\theta_\infty}{2} + (\cdots), \quad (A_x)_{12} = -(A_0)_{12} - \frac{\theta_\infty}{2} + (\cdots).$$

Here the sign  $\pm$  is chosen according to  $(T^{-1}\Lambda_0 T)_{11} = \mp \theta_x/2$ , and  $(\cdots)$  denotes a series of the form  $\sum_{n=1}^{\infty} x^n \sum_{j=0}^{n(\theta)} c_{nj}^* \log^j(\rho x)$  with  $c_{jn}^* \in \mathbb{Q}_{\tilde{\theta}}$ ,  $n(\theta)$  being such that  $n(\theta) = 2n+2$



if  $\theta_0^2 - \theta_x^2 \neq 0$ , and  $= n + 1$  if  $\theta_0^2 - \theta_x^2 = 0$ . Note that

$$-Y_{11} = 1 + \frac{\theta_0 + \theta_x - \theta_\infty + (\cdots)}{2(A_0)_{11} - \theta_x + \theta_\infty + (\cdots)}, \quad -Y_{12} = 1 + \frac{\theta_\infty + (\cdots)}{2(A_0)_{12} + (\cdots)}.$$

If  $\theta_0^2 - \theta_x^2 \neq 0$ , then

$$\begin{aligned} -Y_{11} &= 1 + \frac{4(\theta_0 + \theta_x - \theta_\infty)}{\theta_\infty(\theta_0^2 - \theta_x^2)} \log^{-2}(\rho x) + \cdots, \\ -Y_{12} &= 1 + \frac{4\theta_\infty}{\theta_\infty(\theta_0^2 - \theta_x^2)} \log^{-2}(\rho x) + \cdots; \end{aligned}$$

and if  $\theta_0^2 = \theta_x^2 \neq 0$ , then

$$\begin{aligned} -Y_{11} &= 1 \mp \frac{\theta_0 + \theta_x - \theta_\infty}{\theta_x \theta_\infty} \log^{-1}(\rho x) \left( 1 + \frac{\theta_x \mp \theta_\infty}{\theta_x \theta_\infty} \log^{-1}(\rho x) + \cdots \right), \\ -Y_{12} &= 1 \mp \frac{\theta_\infty}{\theta_x \theta_\infty} \log^{-1}(\rho x) \left( 1 + \frac{2\theta_x \mp \theta_\infty}{\theta_x \theta_\infty} \log^{-1}(\rho x) + \cdots \right). \end{aligned}$$

From these formulas, we derive  $y_{\text{ilog}}(\rho, x)$  and  $y_{\text{ilog}}^\pm(\rho, x)$ . In the case where  $\theta_0^2 - \theta_x^2 \neq 0$ , apparently, there exist two kinds of inverse logarithmic solutions depending on the sign of  $\mp \theta_x$ , but by the following proposition verified later we may replace  $\rho$  suitably to derive  $y_{\text{ilog}}(\rho, x)$  as in (1) independent of the sign; indeed  $A_0^*(\rho, x)$  in the proposition has entries as follows:

$$\begin{aligned} (A_0^*)_{11} &= \frac{\theta_\infty}{8}(\theta_0^2 - \theta_x^2) \log^2(\rho x) - \frac{1}{4}(\theta_0^2 - \theta_x^2) \log(\rho x) - \frac{\theta_\infty \theta_x^2}{2}(\theta_0^2 - \theta_x^2)^{-1} - \frac{\theta_\infty}{2} + \cdots, \\ (A_0^*)_{12} &= (A_0^*)_{11} - \frac{1}{4}(\theta_0^2 - \theta_x^2) \log(\rho x) + \frac{1}{2\theta_\infty}(\theta_0^2 - \theta_x^2) + \cdots. \end{aligned}$$

**Proposition 5.1.** *Suppose that  $\theta_0^2 - \theta_x^2 \neq 0$ . Let  $A_0^-(\rho, x)$  denote  $A_0(\rho, x)$  in the case where  $(T^{-1}\Lambda_0 T)_{11} = \theta_x/2$ . Then  $A_0^*(\rho, x) := A_0^-(\rho \exp(-2\theta_x(\theta_0^2 - \theta_x^2)^{-1}), x) = A_0(\rho \exp(\pm 2\theta_x(\theta_0^2 - \theta_x^2)^{-1}), x)$  is represented by*

$$\begin{aligned} &\frac{1}{4}T \begin{pmatrix} (\theta_0^2 - \theta_x^2) \log(\rho x) & 4 - (\theta_0^2 - \theta_x^2) \log^2(\rho x) + 4\theta_x^2(\theta_0^2 - \theta_x^2)^{-1} \\ \theta_0^2 - \theta_x^2 & -(\theta_0^2 - \theta_x^2) \log(\rho x) \end{pmatrix} T^{-1} \\ &+ \sum_{n=1}^{\infty} x^n \sum_{j=0}^{n(\theta)} A_{jn}^* \log^j(\rho x) \end{aligned}$$

with  $A_{jn}^* \in M_2(\mathbb{Q}_\theta[(\theta_0^2 - \theta_x^2)^{-1}])$ , which is independent of the sign  $\pm$ .

If  $\theta_0 = \theta_x \neq 0$ , from

$$Y_{11}Y_{12} = \frac{(2(A_0)_{11} + \theta_0 + (\cdots))(2(A_0)_{12} + \theta_\infty + (\cdots))}{(2(A_0)_{11} - \theta_x + \theta_\infty + (\cdots))(2(A_0)_{12} + (\cdots))}$$

we derive the expression

$$y_{\text{ilog}}^+(\rho, x) = 1 - \frac{2}{\theta_\infty} \log^{-1}(\rho x) + \sum_{n=1}^{\infty} x^n \sum_{j \geq 0} c_{jn}^+ \log^{n-j}(\rho x).$$

Let us suppose that  $\theta_\infty = 0$ . Then we use Lemma 4.2, (2). If  $\Lambda = \Delta$ ,

$$\begin{aligned}(A_0)_{11} &= \mp \frac{\theta_x}{2} + \frac{1}{4}(\theta_0^2 - \theta_x^2) \log(\rho x) + (\cdots)_{11}, \\ (A_0)_{12} &= 1 \pm \theta_x \log(\rho x) - \frac{1}{4}(\theta_0^2 - \theta_x^2) \log^2(\rho x) + (\cdots)_{12}, \\ (A_x)_{11} &= -(A_0)_{11} + (\cdots)_{11}, \quad (A_x)_{12} = -(A_0)_{12} + 1 + (\cdots)_{12},\end{aligned}$$

and hence, under the condition  $\theta_0^2 - \theta_x^2 \neq 0$  or  $\theta_x \neq 0$ ,

$$-Y_{11} = 1 + \frac{\theta_0 + \theta_x + (\cdots)_{11}}{2(A_0)_{11} - \theta_x + (\cdots)_{11}}, \quad -Y_{12} = 1 - \frac{1 + (\cdots)_{12}}{(A_0)_{12} + (\cdots)_{12}},$$

where  $(\cdots)_{11}$  (respectively,  $(\cdots)_{12}$ ) denotes the sum of  $c_{nj}^* x^n \log^j(\rho x)$  for  $n \geq 1$  and for  $0 \leq j \leq n(\theta) - 1$  (respectively,  $0 \leq j \leq n(\theta)$ ). If  $\Lambda = \Delta_-$ ,

$$\begin{aligned}(A_0)_{11} &= \frac{1}{4}(\theta_0^2 - \theta_x^2) \pm \frac{\theta_x}{2} - \frac{1}{4}(\theta_0^2 - \theta_x^2) \log(\rho x) + (\cdots)_{11}, \\ (A_0)_{12} &= \frac{1}{4}(\theta_0^2 - \theta_x^2) + (\cdots)_{12}, \\ (A_x)_{11} &= -(A_0)_{11} + (\cdots)_{11}, \quad (A_x)_{12} = -(A_0)_{12} + (\cdots)_{12},\end{aligned}$$

and hence, under the condition  $\theta_0^2 - \theta_x^2 \neq 0$ ,

$$-Y_{11} = 1 + \frac{\theta_0 + \theta_x + (\cdots)_{11}}{2(A_0)_{11} - \theta_x + (\cdots)_{11}}, \quad -Y_{12} = 1 + (\cdots)_{12},$$

where  $(\cdots)_{11}$  (respectively,  $(\cdots)_{12}$ ) denotes the sum of  $c_{nj}^* x^n \log^j(\rho x)$  for  $n \geq 1$  and for  $0 \leq j \leq n(\theta) - 1$  (respectively,  $0 \leq j \leq n(\theta) - 2$ ). The solutions  $y_{\text{ilog}}^{(1)}(\rho, x)$  and  $y_{\text{ilog}}^{(2)}(\rho, x)$  in (3) are derived from the formulas above for  $\Lambda = \Delta$  and for  $\Lambda = \Delta_-$ , respectively. In case  $\theta_0^2 - \theta_x^2 \neq 0$ , we use Proposition 5.1. If  $\Lambda = \Delta$ ,  $\theta_0 + \theta_x = 0$  and  $(A_0)_{11} = -\theta_x/2 + (\cdots)_{11}$ , then  $y_{\text{ilog}}^{(1)}(\rho, x)$  follows. Thus we obtain the expressions in Theorem 2.4.

*Proof of Proposition 5.1.* Note that  $t^\Lambda \Lambda_0 t^{-\Lambda} = T \hat{\Lambda}_0(\tau) T^{-1}$ ,  $t^\Lambda \Lambda_x t^{-\Lambda} = T(\Delta - \hat{\Lambda}_0(\tau)) T^{-1}$ , where  $\tau = t \exp(\mp 2\theta_x(\theta_0^2 - \theta_x^2)^{-1})$  and

$$\begin{aligned}\hat{\Lambda}_0(\tau) &:= t^\Delta T^{-1} \Lambda_0 T t^{-\Delta} \\ &= \Delta + \frac{1}{4} \begin{pmatrix} (\theta_0^2 - \theta_x^2) \log \tau & -(\theta_0^2 - \theta_x^2) \log^2 \tau + 4\theta_x^2(\theta_0^2 - \theta_x^2)^{-1} \\ \theta_0^2 - \theta_x^2 & -(\theta_0^2 - \theta_x^2) \log \tau \end{pmatrix} \\ &= \tau^\Delta \begin{pmatrix} 0 & 1 + \theta_x^2(\theta_0^2 - \theta_x^2)^{-1} \\ (\theta_0^2 - \theta_x^2)/4 & 0 \end{pmatrix} \tau^{-\Delta}.\end{aligned}$$

Putting  $t = c_\pm \tau$ ,  $\kappa' = c_\pm \kappa$  with  $c_\pm = \exp(\pm 2\theta_x(\theta_0^2 - \theta_x^2)^{-1})$  in (3.3), we have

$$\begin{aligned}U_\infty^{(0)} &= U_0^{(0)} \equiv 0, \\ U_\infty^{(\nu+1)} &= \mathcal{I} \left[ \kappa' \tau [J/2, T(\Delta - \hat{\Lambda}_0(\tau)) T^{-1} - (c_\pm \tau)^\Lambda U_0^{(\nu)} (c_\pm \tau)^{-\Lambda} + U_\infty^{(\nu)}] \right], \\ (c_\pm \tau)^\Lambda U_0^{(\nu+1)} (c_\pm \tau)^{-\Lambda} &= \tau^\Lambda \mathcal{I} \left[ \tau^{-\Lambda} [U_\infty^{(\nu+1)}, T \hat{\Lambda}_0(\tau) T^{-1} + (c_\pm \tau)^\Lambda U_0^{(\nu)} (c_\pm \tau)^{-\Lambda}] \tau^\Lambda \right] \tau^{-\Lambda},\end{aligned}$$

since

$$[t^{-\Lambda}U_{\infty}^{(\nu+1)}t^{\Lambda}, \Lambda_0 + U_0^{(\nu)}] = t^{-\Lambda}[U_{\infty}^{(\nu+1)}, t^{\Lambda}(\Lambda_0 + U_0^{(\nu)})t^{-\Lambda}]t^{\Lambda}.$$

Using this new recursive relation, we may inductively show that, for every integer  $\nu$ ,  $((c_{\pm}\tau)^{\Lambda}U_0^{(\nu)}(c_{\pm}\tau)^{-\Lambda}, U_{\infty}^{(\nu)})$  does not depend on the choice of the sign  $\pm$ . Write  $\hat{U}_0^{\infty}(\kappa', \tau) := \lim_{\nu \rightarrow \infty} (c_{\pm}\tau)^{\Lambda}U_0^{(\nu)}(c_{\pm}\tau)^{-\Lambda}$ . Setting  $A_0^*(\rho, x) = T\hat{\Lambda}_0(\rho x)T^{-1} + \hat{U}_0^{\infty}(1/\rho, \rho x)$ , which is equal to  $A_0(c_{\pm}\rho, x) = A_0^-(c_{-}\rho, x)$ , we arrive at the conclusion of Proposition 5.1.  $\square$

**5.4. Proof of Theorem 2.5.** Suppose that  $\theta_{\infty} = 0$ . Let  $\Lambda_0 = -\Lambda_x$  be as in Lemma 4.3. Then  $U_0$  and  $U_{\infty}$  such that  $A_0 = \Lambda_0 + U_0$ ,  $A_0 + A_x = U_{\infty}$  satisfy

$$(5.8) \quad \begin{aligned} t \frac{dU_0}{dt} &= [U_{\infty}, \Lambda_0 + U_0], \\ t \frac{dU_{\infty}}{dt} &= t[J, -\Lambda_0 - U_0 + U_{\infty}] \end{aligned}$$

with  $t = x/2$  (cf. (3.1), (3.2)).

**Proposition 5.2.** *System (5.8) admits a solution  $(U_0, U_{\infty}) = (U_0^*(t), U_{\infty}^*(t))$  with*

$$U_0^*(t) = \sum_{j=1}^{\infty} U_j^0 t^j, \quad U_{\infty}^*(t) = \sum_{j=1}^{\infty} U_j^{\infty} t^j$$

*holomorphic around  $t = 0$ . Here  $U_j^0, U_j^{\infty} \in M_2(\mathbb{Q}[\theta_0, a])$ , and  $(U_{\infty}^*(t))_{11} = (U_{\infty}^*(t))_{22} \equiv 0$ .*

*Proof.* For  $V = \sum_{j=0}^{\infty} V_j t^j \in M_2(\mathbb{Q}[\theta_0, a])[[t]]$  and for  $n \in \mathbb{N}$  write  $V = O(t^n)$  if  $V = \sum_{j=n}^{\infty} V_j t^j$ . If  $V = O(t)$ , we may set  $\mathcal{I}[V] := \sum_{j=1}^{\infty} (V_j/j) t^j \in M_2(\mathbb{Q}[\theta_0, a])[[t]]$ , and then  $t(d/dt)\mathcal{I}[V] = V$ . By induction on  $n$  we may define  $U_0^{(n)}, U_{\infty}^{(n)} \in M_2(\mathbb{Q}[\theta_0, a])[[t]]$  ( $n \geq 1$ ) by

$$\begin{aligned} U_{\infty}^{(0)} &\equiv 0, \quad U_0^{(0)} \equiv 0, \\ U_{\infty}^{(n+1)} &= \mathcal{I}[t[J, -\Lambda_0 - U_0^{(n)} + U_{\infty}^{(n)}]], \quad U_0^{(n+1)} = \mathcal{I}[[U_{\infty}^{(n+1)}, \Lambda_0 + U_0^{(n)}]]. \end{aligned}$$

Indeed, supposing  $U_{\infty}^{(n)} = O(t)$  and  $U_0^{(n)} = O(t)$  we easily show  $U_{\infty}^{(n+1)} = O(t)$  and  $U_0^{(n+1)} = O(t)$ . Furthermore, since

$$\begin{aligned} U_{\infty}^{(n+1)} - U_{\infty}^{(n)} &= \mathcal{I}[t[J, -(U_0^{(n)} - U_0^{(n-1)}) + (U_{\infty}^{(n)} - U_{\infty}^{(n-1)})]], \\ U_0^{(n+1)} - U_0^{(n)} &= \mathcal{I}[[U_{\infty}^{(n+1)} - U_{\infty}^{(n)}, \Lambda_0 + U_0^{(n)}] + [U_{\infty}^{(n)}, U_0^{(n)} - U_0^{(n-1)}]], \\ U_{\infty}^{(1)} - U_{\infty}^{(0)} &= O(t), \quad U_0^{(1)} - U_0^{(0)} = O(t), \end{aligned}$$

we have  $U_{\infty}^{(n)} - U_{\infty}^{(n-1)} = O(t^n)$ ,  $U_0^{(n)} - U_0^{(n-1)} = O(t^n)$  for  $n \geq 1$ . Hence  $U_{\infty}^*(t) := \lim_{n \rightarrow \infty} U_{\infty}^{(n)}$ ,  $U_0^*(t) := \lim_{n \rightarrow \infty} U_0^{(n)}$  are in  $M_2(\mathbb{Q}[\theta_0, a])[[t]]$ , and  $(U_0^*(t), U_{\infty}^*(t))$  formally solves (5.8). By [5, Theorem A],  $U_0^*(t)$  and  $U_{\infty}^*(t)$  are convergent around  $t = 0$ . It is easy to see that the diagonal part of  $U_{\infty}^*(t)$  vanishes identically.  $\square$

From the relations for  $n = 0, 1$  in the proof above we have

$$\begin{aligned} (U_1^{\infty})_{11} &= 0, \quad (U_1^{\infty})_{12} = 2, \quad (U_1^0)_{11} = 4a(\theta_0 + a), \quad (U_1^0)_{12} = -2(\theta_0 + 2a), \\ (U_2^{\infty})_{11} &= 0, \quad (U_2^{\infty})_{12} = 2(\theta_0 + 2a + 1). \end{aligned}$$

Note that  $(A_0, A_x) = (\Lambda_0 + U_0^*(x/2), -\Lambda_0 - U_0^*(x/2) + U_\infty^*(x/2))$  solves (2.10). Here

$$(A_0)_{11} = -(A_x)_{11} = \theta_0/2 + a + 2a(\theta_0 + a)x + \cdots, \quad (A_0)_{12} = -1 - (\theta_0 + 2a)x + \cdots, \\ (A_x)_{12} = -(A_0)_{12} + x + (1/2)(\theta_0 + 2a + 1)x^2 + \cdots,$$

and hence  $-Y_{12} = 1 + x + (1/2)(1 - \theta_0 - 2a)x^2 + \cdots$ . If  $\theta_0 = -\theta_x$ , then  $-Y_{11} = 1$ ; and if  $\theta_0 = \theta_x$ , then  $-Y_{11} = (a + \theta_0)a^{-1}(1 - 2\theta_0x + \cdots)$ . From these series we obtain  $y_{\text{Taylor}}^\pm(a, x)$ .

## 6. PROOF OF THEOREM 2.6

Recall the Bäcklund transformation for (V) by Gromak [4] (see also [5, §39], [16]).

**Lemma 6.1.** *Let  $y$  be a given solution of (V) and let  $\pi$  be the substitution defined by (2.4), that is,*

$$\pi : (\theta_0 - \theta_x, \theta_0 + \theta_x, \theta_\infty) \mapsto (1 - \theta_\infty, 1 - \theta_0 + \theta_x, \theta_0 + \theta_x - 1).$$

Set

$$\hat{y} = \hat{B}(y) := 1 - \frac{2xy}{xy' - (\theta_0 - \theta_x + \theta_\infty)y^2/2 + (\theta_\infty + x)y + (\theta_0 - \theta_x - \theta_\infty)/2} \\ = 1 - \frac{2xy}{2(y-1)^2(A_x)_{11} + \theta_x y^2 + 2xy - \theta_x}.$$

Then  $\hat{y}^\pi$ , which is the result of application of  $\pi$  to  $\hat{y}$ , solves (V), that is,

$$\hat{y}^\pi = \hat{B}(y)^\pi = B(y^\pi) : \\ = 1 - \frac{2xy^\pi}{x(y^\pi)' - (\theta_0 + \theta_x - \theta_\infty)(y^\pi)^2/2 + (\theta_0 + \theta_x - 1 + x)y^\pi + 1 - (\theta_0 + \theta_x + \theta_\infty)/2}$$

is also a solution of (V).

The second expression of  $\hat{B}(y)$  follows from

$$2(A_x)_{11}(y-1)^2 = xy' - (\theta_0 + \theta_\infty)(y-1)^2 - xy \\ + (1/2)(y-1)((\theta_0 - \theta_x + \theta_\infty)y - (3\theta_0 + \theta_x + \theta_\infty))$$

(cf. [2, (1.2), (1.3), (1.4)], [13]).

Concerning the uniqueness of a solution of (V) near  $x = 0$  we have

**Lemma 6.2.** *Let  $\sigma, \rho_0 \in \mathbb{C} \setminus \{0\}$  with  $\text{Im } \sigma \neq 0$ . Let  $L(r_0, \omega)_\sigma$  be the curve defined by (2.5). If  $0 < \omega < 1$  (respectively,  $1 < \omega < 2$ ), then a solution of (V) such that*

$$y(x) = 1 + \rho_0 x^{-\sigma}(1 + o(1)) \quad (\text{respectively, } = 1 + \rho_0 x^\sigma(1 + o(1)))$$

as  $x \rightarrow 0$  along  $L(r_0, \omega)_\sigma$  is uniquely determined.

*Proof.* By  $y = \tanh^2(u/2)$ , (V) is changed into  $x(xu')' = f(x, e^{-u}, xe^u)$  with

$$f(x, e^{-u}, xe^u) = \frac{1}{8} \left( (\theta_0 - \theta_x + \theta_\infty)^2 \frac{\sinh(u/2)}{\cosh^3(u/2)} - (\theta_0 - \theta_x - \theta_\infty)^2 \frac{\cosh(u/2)}{\sinh^3(u/2)} \right)$$

$$+ \frac{1}{2}(1 - \theta_0 - \theta_x)x \sinh u + \frac{x^2}{8} \sinh(2u),$$

where  $f(x, \xi, \eta)$  is holomorphic around  $x = \xi = \eta = 0$  and  $f(0, 0, 0) = 0$ . Note that  $|\rho_0 x^{1+\sigma}| = O(|x|^\omega)$  along  $L(r_0, \omega)_\sigma$ . Suppose that  $1 < \omega < 2$  and that  $y(x) = 1 + \rho_0 x^\sigma(1 + o(1)) = 1 + O(|x|^{\omega-1})$  along  $L(r_0, \omega)_\sigma$ . Let  $u(x)$  and  $v(x)$  be such that  $u(x) = -\sigma \log x - \log(-\rho_0/4) - v(x)$  with  $y(x) = \tanh^2(u(x)/2) = 1 - 4e^{-u(x)}(1 + O(e^{-u(x)}))$ . They satisfy  $(y(x) - 1)(\rho_0 x^\sigma)^{-1} = e^{v(x)}(1 + O(e^{-u(x)})) = e^{v(x)}(1 + O(|x|^{\omega-1}))$ , which implies  $v(x) = o(1)$  along  $L(r_0, \omega)_\sigma$ . Then  $v = v(x)$  solves

$$x(xv')' = g(x, \rho_0 x^\sigma, \rho_0^{-1} x^{1-\sigma}, v),$$

where  $g(x, \xi, \eta, v) = f(x, \xi e^v, \eta e^{-v})$ . The function  $g(x, \xi, \eta, v)$  is holomorphic around  $x = \xi = \eta = v = 0$  and satisfies  $g(x, \xi, \eta, v) = O(|x| + |\xi| + |\eta|)$  and  $g(x, \xi, \eta, \tilde{v}) - g(x, \xi, \eta, v) = O(|x| + |\xi| + |\eta|)|\tilde{v} - v|$  if  $|v|$  and  $|\tilde{v}|$  are small. For  $x, x_0 \in L(r_0, \omega)_\sigma$

$$x_0 v'(x_0) - x v'(x) = \int_{L(x_0) \setminus L(x)} g(t, \rho_0 t^\sigma, \rho_0^{-1} t^{1-\sigma}, v(t)) \frac{dt}{t},$$

where  $L(x) \subset L(r_0, \omega)_\sigma$  is a curve joining 0 to  $x$  given by  $t = \tau e^{i\theta(\tau)}$ ,  $\tau = |t|$ ,  $0 < \tau \leq |x|$  with  $\theta(\tau) = ((1 + \operatorname{Re} \sigma - \omega) \log \tau - r_0) / \operatorname{Im} \sigma$ . Observing that  $dt/d\tau = O(1)$  and  $g(t, \rho_0 t^\sigma, \rho_0^{-1} t^{1-\sigma}, v(t)) = O(\tau^{\omega-1} + \tau^{2-\omega})$  along  $L(r_0, \omega)_\sigma$ , we have  $xv'(x) \rightarrow c_0$  as  $x \rightarrow 0$  for some  $c_0 \in \mathbb{C}$ . Since  $v(x) = o(1)$ , we have  $c_0 = 0$ , and hence

$$v(x) = \int_{L(x)} \int_{L(s)} g(t, \rho_0 t^\sigma, \rho_0^{-1} t^{1-\sigma}, v(t)) \frac{dt}{t} \frac{ds}{s}.$$

If  $v_1(x), v_2(x) = o(1)$  are solutions of this equation, then  $\phi(x) = \sup_{t \in L(x)} |v_2(t) - v_1(t)|$  satisfies  $\phi(x) = O(|x|^{\omega-1} + |x|^{2-\omega})\phi(x)$ , which implies the uniqueness of  $y(x)$ .  $\square$

By Remark 2.1,  $\Omega_{\sigma, \rho}^-(\varepsilon_0)$  and  $\Omega_{\sigma, \rho}^+(\varepsilon_0)$  are spanned by  $L(r_0, \omega)_\sigma$  with  $0 < \omega < 1$  and  $1 < \omega < 2$ , respectively. Hence

$$y(\sigma, \rho, x) = \begin{cases} y_-(\sigma, \rho, x) \sim 1 + c(-\sigma)\rho^{-1}x^{-\sigma} & \text{on } L(r_0, \omega)_\sigma \text{ with } 0 < \omega < 1, \\ y_+(\sigma, \rho, x) \sim 1 + c(\sigma)\rho x^\sigma & \text{on } L(r_0, \omega)_\sigma \text{ with } 1 < \omega < 2 \end{cases}$$

with  $c(\sigma)$  given by (2.3). To  $y(\sigma, \rho, x)$ ,  $y_\pm(\sigma, \rho, x)$  and  $c(\sigma)$ , apply  $\pi$  of Lemma 6.1, and denote the results by  $y^\pi(\sigma, \rho, x)$ ,  $y_\pm^\pi(\sigma, \rho, x)$  and  $\tilde{c}(\sigma) := c^\pi(\sigma)$ . Then the results of the Bäcklund transformation  $y^*(\sigma, \rho, x) := B(y^\pi(\sigma, \rho, x))$ ,  $y_\pm^*(\sigma, \rho, x) := B(y_\pm^\pi(\sigma, \rho, x))$  also solve (V) and satisfy

$$y^*(\sigma, \rho, x) = \begin{cases} y_-^*(\sigma, \rho, x) \sim 1 + \frac{2\rho x^{1+\sigma}}{(1 - \theta_\infty + \sigma)\tilde{c}(-\sigma)} & \text{on } L(r_0, \omega)_\sigma \text{ with } 0 < \omega < 1, \\ y_+^*(\sigma, \rho, x) \sim 1 + \frac{2\rho^{-1}x^{1-\sigma}}{(1 - \theta_\infty - \sigma)\tilde{c}(\sigma)} & \text{on } L(r_0, \omega)_\sigma \text{ with } 1 < \omega < 2. \end{cases}$$

From the facts above, for every  $\nu \in \mathbb{Z}$  we derive the following expressions along the curve  $L(r_0, \omega)_\sigma = L(r_0, \omega - 2\nu)_{\sigma-2\nu} = L(r_0, \omega - 2\nu + 1)_{\sigma-2\nu+1}$ :

(1) if  $2\nu < \omega < 2\nu + 1$ ,

$$y(\sigma - 2\nu, \rho, x) \sim 1 + c(2\nu - \sigma)\rho^{-1}x^{2\nu-\sigma};$$

(2) if  $2\nu + 1 < \omega < 2\nu + 2$ ,

$$y(\sigma - 2\nu, \rho, x) \sim 1 + c(\sigma - 2\nu)\rho x^{\sigma-2\nu};$$

(3) if  $2\nu - 1 < \omega < 2\nu$ ,

$$y^*(\sigma - 2\nu + 1, \hat{\rho}, x) \sim 1 + \frac{2\hat{\rho}x^{\sigma-2\nu+2}}{(2 - \theta_\infty - 2\nu + \sigma)\tilde{c}(2\nu - 1 - \sigma)};$$

(4) if  $2\nu < \omega < 2\nu + 1$ ,

$$y^*(\sigma - 2\nu + 1, \hat{\rho}, x) \sim 1 + \frac{2\hat{\rho}^{-1}x^{2\nu-\sigma}}{(2\nu - \theta_\infty - \sigma)\tilde{c}(\sigma - 2\nu + 1)}.$$

By Lemma 6.2, matching (1) with (4), we derive

$$(6.1) \quad y(\sigma - 2\nu, \rho, x) \equiv y^*(\sigma - 2\nu + 1, \hat{\rho}, x)$$

if

$$c(2\nu - \sigma)\rho^{-1} = \frac{2\hat{\rho}^{-1}}{(2\nu - \theta_\infty - \sigma)\tilde{c}(\sigma - 2\nu + 1)}.$$

Similarly, from (2) and (3) we obtain

$$(6.2) \quad y(\sigma - 2\nu, \rho, x) \equiv y^*(\sigma - 2\nu - 1, \hat{\rho}, x)$$

if

$$c(\sigma - 2\nu)\rho = \frac{2\hat{\rho}}{(\sigma - \theta_\infty - 2\nu)\tilde{c}(2\nu + 1 - \sigma)}.$$

From these relations Theorem 2.6 follows.

*Remark 6.1.* For  $y^*(\sigma, \rho, x)$  we have

$$y^*(\sigma - 2\nu + 1, \rho, x) \equiv y^*(\sigma - 2\nu - 1, \gamma^*(\sigma, \nu)\rho, x),$$

where

$$\gamma^*(\sigma, \nu) = \frac{1}{4}(\sigma - \theta_\infty - 2\nu)(2\nu - \theta_\infty - \sigma)c(\sigma - 2\nu)c(2\nu - \sigma)\tilde{c}(2\nu + 1 - \sigma)\tilde{c}(\sigma - 2\nu + 1).$$

## 7. PROOFS OF THEOREMS 2.2, AND 2.7 THROUGH 2.12

**7.1. Proofs of Theorems 2.7 through 2.12.** By (5.5) and (5.6) we write  $Y_{11}$  and  $Y_{12}$  as follows:

$$-Y_{11} = \frac{F(x)\Phi_1(x) + O(x)}{G(x)\Psi_1(x) + O(x)}, \quad -Y_{12} = \frac{G(x)\Phi_2(x) + O(x)}{F(x)\Psi_2(x) + O(x)}$$

in  $D_{\text{even}}(\sigma, \rho, 0) = \{x \in \mathcal{R}(\mathbb{C} \setminus \{0\}); |x| < \varepsilon_0, \varepsilon_0 < |\rho x^\sigma| < \varepsilon_0^{-1}\}$ , where

$$F(x) = (\sigma - \theta_\infty)(\sigma + \theta_0 - \theta_x)(\rho x^\sigma)^{1/2} - (\sigma + \theta_\infty)(\sigma - \theta_0 + \theta_x)(\rho x^\sigma)^{-1/2},$$

$$G(x) = (\sigma - \theta_\infty)(\rho x^\sigma)^{1/2} + (\sigma + \theta_\infty)(\rho x^\sigma)^{-1/2},$$

$$\Phi_1(x) = (\sigma - \theta_0 - \theta_x)(\rho x^\sigma)^{1/2} + (\sigma + \theta_0 + \theta_x)(\rho x^\sigma)^{-1/2},$$

$$\Psi_1(x) = ((\sigma - \theta_x)^2 - \theta_0^2)(\rho x^\sigma)^{1/2} - ((\sigma + \theta_x)^2 - \theta_0^2)(\rho x^\sigma)^{-1/2},$$

$$\begin{aligned}\Phi_2(x) &= (\sigma - \theta_\infty)((\sigma - \theta_x)^2 - \theta_0^2)(\rho x^\sigma)^{1/2} + (\sigma + \theta_\infty)((\sigma + \theta_x)^2 - \theta_0^2)(\rho x^\sigma)^{-1/2}, \\ \Psi_2(x) &= (\sigma - \theta_\infty)(\sigma - \theta_0 - \theta_x)(\rho x^\sigma)^{1/2} - (\sigma + \theta_\infty)(\sigma + \theta_0 + \theta_x)(\rho x^\sigma)^{-1/2}.\end{aligned}$$

Let us seek the zeros of

$$\Phi_1(x) = (\sigma - \theta_0 - \theta_x)\rho(\rho x^\sigma)^{-1/2}(x^\sigma - \xi_0), \quad \xi_0 = -\frac{\sigma + \theta_0 + \theta_x}{\sigma - \theta_0 - \theta_x}\rho^{-1}.$$

Set  $r_0 = \log |\xi_0|$  and  $\mu_0 = \arg \xi_0$ . Then we have

$$x^\sigma - \xi_0 = -2i \exp((\sigma \log x + r_0 + i\mu_0)/2) \sin((i\sigma \log x + \mu_0 - ir_0)/2),$$

in which, along  $L(r_0, 1)_\sigma$ ,

$$\frac{1}{2}(i\sigma \log x + \mu_0 - ir_0) = -\frac{1}{2\operatorname{Im} \sigma}(|\sigma|^2 \log |x| - r_0 \operatorname{Re} \sigma - \mu_0 \operatorname{Im} \sigma).$$

Hence  $\Phi_1(x)$  has zeros  $x_n$ ,  $n \in \mathbb{N}$  such that  $|\sigma|^2 \log |x_n| - r_0 \operatorname{Re} \sigma - \mu_0 \operatorname{Im} \sigma = -2\pi |\operatorname{Im} \sigma| n$  on  $L(r_0, 1)_\sigma$ . Under the supposition of Theorem 2.7 the ratio of  $\Phi_1(x)$  to any of the other five functions is not a constant, and then  $F(x_n), \dots, \Psi_2(x_n)$  other than  $\Phi_1(x_n)$  are nonzero numbers independent of  $n$ , since  $x_n^\sigma = \xi_0$ . Using Rouché's theorem, by the same argument as that of [19, Section 2.2.2] we may prove the existence of the sequence of zeros  $\{x_n^0\}_{n \in \mathbb{N}}$  in Theorem 2.7. The other sequence  $\{\hat{x}_n^0\}_{n \in \mathbb{N}}$  is obtained from zeros of  $\Phi_2(x)$ . Theorem 2.8 is proved by using  $\Psi_1(x)$  and  $\Psi_2(x)$ . Theorem 2.11 is also proved by using (5.7) by the same argument as above. Theorem 2.9 follows from the facts that  $\Phi_1(x)/\Phi_2(x)$  is a constant if  $\theta_0 - \theta_x - \theta_\infty = 0$  and that then every zero of a solution of (V) is double, and from analogous facts on poles.

To prove Theorem 2.10 we use the Bäcklund transformation. Let  $y$  and  $\hat{y} = \hat{B}(y)$  be as in Lemma 6.1. Then

$$\frac{1}{\hat{y}} - 1 = \frac{2xy}{2(y-1)^2(A_x)_{11} + \theta_x(y^2 - 1)}.$$

For  $x \in D_{\text{even}}(\sigma, \rho, 0)$  substitute

$$(7.1) \quad y = y(\sigma, \rho, x) = \frac{\Phi_1(x)\Phi_2(x)F(x)G(x) + O(x)}{\Psi_1(x)\Psi_2(x)F(x)G(x) + O(x)}$$

into the right-hand side. Observing that

$$(A_x)_{11} + \frac{\theta_x}{2} = \frac{1}{8\sigma^2}(Y_{11})_{\text{den}} = -\frac{1}{8\sigma^2}(G(x)\Psi_1(x) + O(x))$$

with  $(Y_{11})_{\text{den}}$  as in (5.5), that  $\Phi_1(x)\Phi_2(x) - \Psi_1(x)\Psi_2(x) = 4\sigma^2(\sigma^2 - (\theta_0 + \theta_x)^2)$ , and that  $(\sigma^2 - (\theta_0 + \theta_x)^2)G(x) - 2\theta_x\Psi_2(x) = \Phi_2(x)$ , we obtain

$$(7.2) \quad 1 - \frac{1}{\hat{y}} = \frac{x(\Phi_1(x)\Phi_2(x)\Psi_1(x)\Psi_2(x)F(x)^2G(x)^2 + O(x))}{2\sigma^2(\sigma^2 - (\theta_0 + \theta_x)^2)\Phi_2(x)\Psi_1(x)F(x)^2G(x)^2 + O(x)}.$$

Under the supposition

$$(7.3) \quad \begin{aligned} &\theta_0(\theta_0 + \theta_x - \theta_\infty)((\theta_0 - \theta_x)^2 - \theta_\infty^2)(\sigma^2 - \theta_\infty^2) \\ &\quad \times (\sigma^2 - (\theta_0 \pm \theta_x)^2)(\theta_0^2 - \theta_x^2 + \sigma^2 - 2\theta_\infty\theta_0) \neq 0, \end{aligned}$$

which is a part of those of Theorems 2.7 and 2.8, the ratio of  $\Phi_1(x)$  or of  $\Psi_2(x)$  to any of the other five functions is not a constant and its zeros have the same property as that of  $x_n$  above. Setting  $(\sigma, \rho) = (\hat{\sigma}, \hat{\rho})$  in (7.2) and (7.3), we apply  $\pi$  of Lemma 6.1 to them. Then (7.3) becomes (2.8) with  $\sigma = \hat{\sigma} - 1$ . From the result of the application of  $\pi$  to (7.2) with  $(\hat{\sigma}, \hat{\rho})$  we conclude that  $y^*(\hat{\sigma}, \hat{\rho}, x) = B(y^\pi(\hat{\sigma}, \hat{\rho}, x)) = \hat{y}^\pi(\hat{\sigma}, \hat{\rho}, x)$  solving (V) has 1-points near the zeros of  $\Phi_1^\pi(x)$  and  $\Psi_2^\pi(x)$ . By (6.1) with  $\nu = 0$  we have  $y(\sigma, \rho, x) \equiv y^*(\hat{\sigma}, \hat{\rho}, x)$  if

$$\hat{\sigma} = \sigma + 1, \quad \hat{\rho}^{-1} = -(\theta_\infty + \sigma)\tilde{c}(\sigma + 1)c(-\sigma)\rho^{-1}/2.$$

Observing that  $\Phi_1^\pi(x)$  (respectively,  $\Psi_2^\pi(x)$ ) vanishes if

$$x^{\hat{\sigma}} = \xi_1 = -\frac{\hat{\sigma} + 1 - \theta_0 + \theta_x}{\hat{\sigma} - 1 + \theta_0 - \theta_x}\hat{\rho}^{-1} \quad \left( \text{respectively, } x^{\hat{\sigma}} = \hat{\xi}_1 = -\frac{\hat{\sigma} - 1 + \theta_0 + \theta_x}{\hat{\sigma} + 1 - \theta_0 - \theta_x}\hat{\xi}_1 \right),$$

and that  $L(r_1, 1)_{\hat{\sigma}} = L(r_1, 1)_{\sigma+1} = L(r_1, 0)_\sigma$  with  $r_1 = \log|\xi_1|$ , we obtain Theorem 2.10. By the expressions of  $y_{\sigma_0}(\rho, x)$  in  $\Omega^-(\sigma_0, \varepsilon_0)$  of Theorem 2.3,  $y_{\sigma_0}(\rho, x) \sim 1 + c^*(\sigma_0)\rho^{-1}x^{-\sigma_0}$  on  $L(r_0, \omega)_{\sigma_0}$  with  $0 < \omega < 1$  in each case,  $r_0$  being a fixed number. On the other hand

$$y^*(\sigma_0 + 1, \hat{\rho}, x) = y_+^*(\sigma_0 + 1, \hat{\rho}, x) \sim 1 - \frac{2\hat{\rho}^{-1}x^{-\sigma_0}}{(\theta_\infty + \sigma_0)\tilde{c}(\sigma_0 + 1)}$$

on  $L(r_0, \omega + 1)_{\sigma_0+1} = L(r_0, \omega)_{\sigma_0}$  with  $0 < \omega < 1$  (cf. Section 6). Hence  $y_{\sigma_0}(\rho, x) \equiv y^*(\sigma_0 + 1, \hat{\rho}, x)$  with  $-(\theta_\infty + \sigma_0)c^*(\sigma_0)\tilde{c}(\sigma_0 + 1)\hat{\rho} = 2\rho$ . Using this fact we may show Theorem 2.12.

**7.2. Proof of Theorem 2.2.** If  $\theta_0 - \theta_x = \theta_\infty = 0$ , then by [18, Theorem 5.6] equation (V) admits a family of solutions given by

$$\tanh^2\left(\left(\frac{1}{2} - \tilde{\sigma}\right)\log x + \frac{1}{2}\log \tilde{\rho} + s_V(\tilde{\sigma}, x, \tilde{\rho}^{-1/2}x^{\tilde{\sigma}}, \tilde{\rho}^{1/2}x^{1-\tilde{\sigma}})\right)$$

with  $\tilde{\sigma} \in \tilde{\Sigma} \subset \mathbb{C} \setminus (\{\tilde{\sigma} \leq 0\} \cup \{\tilde{\sigma} \geq 1\})$ ,  $\tilde{\rho} \in \mathbb{C} \setminus \{0\}$ ,  $\tilde{\Sigma}$  being a bounded domain such that  $\text{dist}(\tilde{\Sigma}, \{\tilde{\sigma} \leq 0\} \cup \{\tilde{\sigma} \geq 1\}) > 0$ . The series

$$s_V(\tilde{\sigma}, x, \xi, \eta) = \sum_{i \geq 1} c_i^0(\tilde{\sigma})x^i + \sum_{i \geq 0, j \geq 1} c_{ij}^1(\tilde{\sigma})x^i\xi^{2j} + \sum_{i \geq 0, j \geq 1} c_{ij}^2(\tilde{\sigma})x^i\eta^{2j}$$

with  $c_i^0(\tilde{\sigma}), c_{ij}^1(\tilde{\sigma}), c_{ij}^2(\tilde{\sigma}) \in \mathbb{Q}[\theta_0](\tilde{\sigma})$  converges if  $|x|, |\xi|, |\eta|$  are sufficiently small. Note that we have replaced  $(\xi, \eta)$  in [18, Theorem 5.6] by  $(\xi^2, \eta^2)$  since, under the condition  $\theta_0 - \theta_x = \theta_\infty = 0$ , the local equation in [18, §5] equivalent to (V) is written in terms of  $F(x, \xi, \eta) = (1/32)(\eta^2 - \xi^2)(4(1 - \theta_0 - \theta_x) + (\eta^2 + \xi^2))$ . Putting  $\sigma = 1 - 2\tilde{\sigma} \in \Sigma_0$ , we have

$$\tanh^2\left(\frac{1}{2}\log(\tilde{\rho}x^\sigma) + s_V((1 - \sigma)/2, x, (\tilde{\rho}^{-1}x^{1-\sigma})^{1/2}, (\tilde{\rho}x^{1+\sigma})^{1/2})\right),$$

which satisfies

$$= \begin{cases} 1 - 4\tilde{\rho}x^\sigma(1 + o(1)) & \text{in } D_+(\sigma, \tilde{\rho}, 0), \\ 1 - 4(\tilde{\rho}x^\sigma)^{-1}(1 + o(1)) & \text{in } D_-(\sigma, \tilde{\rho}, 0). \end{cases}$$

Since  $((\tilde{\rho}^{\pm 1}x^{1\pm\sigma})^{1/2})^{2j} = x^j(\tilde{\rho}x^\sigma)^{\pm j}$ ,  $s_V(\cdots)$  has the form  $\sum_{n=1}^\infty x^n \sum_{j=-n}^n c_{nj}^*(\tilde{\rho}x^\sigma)^j$ . Noting that  $c(\sigma) = -4(2\theta_0 - \sigma)(2\theta_0 + \sigma)^{-1}$ , we compare the behaviour above with that



of  $y(\sigma, \rho, x)$  to obtain (2.1). By Lemma 6.1,  $\hat{y} = \hat{B}(y) = 1 - 2y/(y' + y)$ . Substitution of (2.1) with  $(\sigma, \rho) = (\hat{\sigma}, \hat{\rho})$  yields

$$\hat{y}(\hat{\sigma}, \hat{\rho}, x) = 1 - \frac{2x \sinh(\log(\check{\rho}_- x^{\hat{\sigma}}) + \Sigma_V(x))}{2\hat{\sigma} + 2x\Sigma'_V(x) + x \sinh(\log(\check{\rho}_- x^{\hat{\sigma}}) + \Sigma_V(x))}$$

with  $\check{\rho}_- = (2\theta_0 - \hat{\sigma})(2\theta_0 + \hat{\sigma})^{-1}\hat{\rho}$ , where

$$\Sigma_V(x) = 2s_V((1 - \hat{\sigma})/2, x, (\check{\rho}_-^{-1}x^{1-\hat{\sigma}})^{1/2}, (\check{\rho}_-x^{1+\hat{\sigma}})^{1/2})$$

and  $\Sigma'_V(x) = (d/dx)\Sigma_V(x)$ . Apply  $\pi$  to both sides. Observing that  $\check{\rho} := (\check{\rho}_-)^{\pi} = (2 - \hat{\sigma})(2 + \hat{\sigma})^{-1}\hat{\rho}$ , and that  $y^*(\hat{\sigma}, \hat{\rho}, x) = \hat{y}^{\pi}(\hat{\sigma}, \hat{\rho}, x)$  coincides with  $y(\sigma, \rho, x)$  if  $\hat{\sigma} = \sigma + 1$  and

$$\hat{\rho}^{-1} = -\sigma\tilde{c}(\sigma + 1)c(-\sigma)\frac{\rho^{-1}}{2} = \frac{8\sigma(\sigma + 1)^2}{(\sigma + 2)(2\theta_0 - \sigma)}\rho^{-1},$$

we obtain (2.2).

## 8. MONODROMY DATA FOR THE ISOMONODROMY DEFORMATION

Let  $(A_0(x), A_x(x))$  be the solution in Proposition 3.1 or 3.2 yielding each solution of (V) in Section 2 (cf. Proposition 3.3 and Section 5). The associated linear system (1.1) admits the matrix solution

$$Y(\lambda, x) = (I + O(\lambda^{-1}))e^{(\lambda/2)J}\lambda^{-(\theta_{\infty}/2)J} \quad \lambda \rightarrow \infty, \quad -\pi/2 < \arg \lambda < 3\pi/2$$

having the isomonodromy property. Set  $\tilde{\rho} = \rho_*^{1/\sigma}$  with

$$\rho_* := \frac{(\theta_0 - \theta_x + \sigma)(\theta_0 + \theta_x - \sigma)}{2\sigma(\sigma + \theta_{\infty})}\rho$$

if  $(A_0(x), A_x(x))$  yields  $y(\sigma, \rho, x)$  (cf. (5.4)),  $\tilde{\rho} = \rho^{1/\sigma_0}$  if it yields  $y_{\sigma_0}(\rho, x)$ , and  $\tilde{\rho} = \rho$  if  $\sigma = 0$ . Then by Proposition 3.1 or 3.2,  $A_0$  and  $A_x$  satisfy

$$x^{-\Lambda}A_0x^{\Lambda} \rightarrow \tilde{\Lambda}_0 := \tilde{\rho}^{\Lambda}\Lambda_0\tilde{\rho}^{-\Lambda}, \quad x^{-\Lambda}A_xx^{\Lambda} \rightarrow \tilde{\Lambda}_x := \tilde{\rho}^{\Lambda}\Lambda_x\tilde{\rho}^{-\Lambda}, \quad A_0 + A_x \rightarrow \Lambda$$

as  $x \rightarrow 0$ , the eigenvalues of  $\Lambda$  being  $\pm\sigma/2$  with  $\sigma \in (\mathbb{C} \setminus \mathbb{Z}) \cup \{0\}$ .

Let us apply the argument of [12, §2] to our case. By [17]

$$\hat{Y}(\lambda) := \lim_{x \rightarrow 0} Y(\lambda, x) \quad \text{and} \quad \tilde{Y}(\lambda) := \lim_{x \rightarrow 0} x^{-\Lambda}Y(x\lambda, x)$$

solve

$$(8.1) \quad \frac{d\hat{Y}}{d\lambda} = \left(\frac{\Lambda}{\lambda} + \frac{J}{2}\right)\hat{Y}$$

and

$$(8.2) \quad \frac{d\tilde{Y}}{d\lambda} = \left(\frac{\tilde{\Lambda}_0}{\lambda} + \frac{\tilde{\Lambda}_x}{\lambda - 1}\right)\tilde{Y},$$

respectively. Since  $\sigma \in (\mathbb{C} \setminus \mathbb{Z}) \cup \{0\}$ , we may choose a matrix solution of (8.1) such that

$$\hat{Y}_0(\lambda) = (I + O(\lambda^{-1}))e^{(\lambda/2)J}\lambda^{-(\theta_{\infty}/2)J} \quad \lambda \rightarrow \infty,$$

$$= (I + O(\lambda))\lambda^\Lambda C \quad \lambda \rightarrow 0,$$

and of (8.2) such that

$$\begin{aligned} (8.3) \quad \tilde{Y}_0(\lambda) &= (I + O(\lambda^{-1}))\lambda^\Lambda & \lambda \rightarrow \infty, \\ &= G_0(I + O(\lambda))\lambda^{(\theta_0/2)J}\lambda^{\Delta_0}C^{(0)} & \lambda \rightarrow 0, \\ &= G_x(I + O(\lambda - 1))(\lambda - 1)^{(\theta_x/2)J}(\lambda - 1)^{\Delta_x}C^{(x)} & \lambda \rightarrow 1 \end{aligned}$$

for  $0 < \arg \lambda < \pi$ ,  $0 < \arg(\lambda - 1) < \pi$ , where  $C$ ,  $C^{(0)}$ ,  $C^{(x)}$ ,  $G_0$ ,  $G_x$  are some invertible matrices, and, for each  $\iota = 0, x$ , the matrix  $\Delta_\iota$  equals 0 if  $\theta_\iota \notin \mathbb{Z}$ ,  $\epsilon_\iota \Delta$  if  $\theta_\iota \in \mathbb{N} \cup \{0\}$ , and  $\epsilon_\iota \Delta_-$  if  $-\theta_\iota \in \mathbb{N}$  with  $\epsilon_\iota \in \mathbb{C}$ . Then by the same argument as in the proof of [12, Proposition 2.1]

$$\hat{Y}(\lambda) = \lim_{x \rightarrow 0} Y(\lambda, x) = \hat{Y}_0(\lambda), \quad \tilde{Y}(\lambda) = \lim_{x \rightarrow 0} x^{-\Lambda} Y(x\lambda, x) = \tilde{Y}_0(\lambda)C$$

as long as  $|x^{1+\sigma}|, |x^{1-\sigma}| \rightarrow 0$ , and

$$\begin{aligned} Y(\lambda, x) &= G_0(x)(I + O(\lambda))\lambda^{(\theta_0/2)J}\lambda^{\Delta_0}C^{(0)}C & \lambda \rightarrow 0, \\ &= G_x(x)(I + O(\lambda - x))(\lambda - x)^{(\theta_x/2)J}(\lambda - x)^{\Delta_x}C^{(x)}C & \lambda \rightarrow x, \end{aligned}$$

where  $G_0(x)$  and  $G_x(x)$  are invertible matrices. Therefore the monodromy matrices  $M_0$ ,  $M_x$  defined in Section 2.4 are written as follows:

$$(8.4) \quad M_\iota = \begin{cases} (C^{(\iota)}C)^{-1}e^{\pi i \theta_\iota J}C^{(\iota)}C & \theta_\iota \notin \mathbb{Z}, \\ (-1)^{\theta_\iota}(C^{(\iota)}C)^{-1}e^{2\pi i \Delta_\iota}C^{(\iota)}C & \theta_\iota \in \mathbb{Z} \end{cases}$$

( $\iota = 0, x$ ). Monodromy data for each solution will be computed by using this fact.

*Remark 8.1.* The matrices  $C^{(\iota)}$  ( $\iota = 0, x$ ) are not determined uniquely. Indeed, say, if  $\theta_\iota \notin \mathbb{Z}$ , we may take  $G_\iota(x)D_0^{-1}$ ,  $D_0C^{(\iota)}$  with any invertible diagonal matrix  $D_0$  instead of  $G_\iota(x)$ ,  $C^{(\iota)}$ .

*Remark 8.2.* In our setting of isomonodromy deformation for system (1.1) with  $x \neq 0$ , the monodromy data and  $y$  are the same as those in [2], and the unknown variables corresponding to  $z = z_{AK}$  and  $u = u_{AK}$  in [2] are  $z_{AK}$  and  $x^{-\theta_\infty}u_{AK}$ , respectively.

## 9. CONNECTION FORMULAS FOR THE WHITTAKER AND THE HYPERGEOMETRIC SYSTEMS

**9.1. Whittaker system.** Since  $(A_0(x), A_x(x))$  mentioned in Section 8 is obtained by using  $\Lambda_0$ ,  $\Lambda_x$ ,  $T$ ,  $\Lambda$  in Lemma 4.1 or 4.2, our concern is (8.1) corresponding to such matrices. The connection matrix  $C$  for (8.1) is given by the following.

**Proposition 9.1.** (1) *Suppose that  $\sigma \notin \mathbb{Z}$ . For  $\Lambda$ ,  $T$  as in Lemma 4.1,  $C = TC_\infty$  with*

$$C_\infty = \begin{pmatrix} -\frac{e^{-\pi i(\sigma + \theta_\infty)/2}\Gamma(-\sigma)}{\Gamma(1 - (\sigma - \theta_\infty)/2)} & -\frac{\Gamma(-\sigma)}{\Gamma(1 - (\sigma + \theta_\infty)/2)} \\ -\frac{e^{\pi i(\sigma - \theta_\infty)/2}\Gamma(\sigma)}{\Gamma((\sigma + \theta_\infty)/2)} & \frac{\Gamma(\sigma)}{\Gamma((\sigma - \theta_\infty)/2)} \end{pmatrix}.$$

(2) Suppose that  $\sigma = 0$ . For  $\Lambda$ ,  $T$  as in Lemma 4.2,  $C = TC_\infty$  with  $C_\infty$  given as follows:

(i) if  $\theta_\infty \neq 0$ ,

$$C_\infty = \begin{pmatrix} \frac{e^{-\pi i \theta_\infty / 2} (\psi(1 + \theta_\infty / 2) - 2\psi(1) - \pi i)}{\Gamma(1 + \theta_\infty / 2)} & \frac{\psi(-\theta_\infty / 2) - 2\psi(1)}{\Gamma(1 - \theta_\infty / 2)} \\ \frac{e^{-\pi i \theta_\infty / 2}}{\Gamma(1 + \theta_\infty / 2)} & \frac{1}{\Gamma(1 - \theta_\infty / 2)} \end{pmatrix};$$

(ii) if  $\theta_\infty = 0$  and  $\Lambda = \Delta$ , then  $C_\infty = I - \psi(1)\Delta$ ;

(iii) if  $\theta_\infty = 0$  and  $\Lambda = \Delta_-$ , then  $C_\infty = (1 - \pi i - \psi(1))(I + J)/2 + \Delta + \Delta_-$ .

*Proof.* If  $(\sigma, \theta_\infty) \neq (0, 0)$ , then system (8.1) with  $\Lambda$  as in Lemma 4.1 or 4.2 has the matrix solution  $\hat{Y}(\lambda)$  given by

$$\begin{pmatrix} e^{\pi i (1 - \theta_\infty) / 2} W_{(1 - \theta_\infty) / 2, \sigma / 2}(e^{-\pi i} \lambda) & -\frac{1}{2}(\sigma - \theta_\infty) W_{(-1 + \theta_\infty) / 2, \sigma / 2}(\lambda) \\ -\frac{1}{2}(\sigma + \theta_\infty) e^{\pi i (1 - \theta_\infty) / 2} W_{-(1 + \theta_\infty) / 2, \sigma / 2}(e^{-\pi i} \lambda) & W_{(1 + \theta_\infty) / 2, \sigma / 2}(\lambda) \end{pmatrix} \lambda^{-1/2}$$

(cf. [12, (3.10)]), which behaves as

$$\hat{Y}(\lambda) = (I + O(\lambda^{-1})) e^{(\lambda/2)J} \lambda^{-(\theta_\infty/2)J} \quad \lambda \rightarrow \infty, \quad 0 < \arg \lambda < \pi.$$

Here  $W_{\kappa, \mu}(z)$  is the Whittaker function such that  $W_{\kappa, \mu}(z) \sim e^{-z/2} z^\kappa$ ,  $|\arg z| < \pi$ . If  $\sigma \notin \mathbb{Z}$ , we have, around  $\lambda = 0$ ,

$$\hat{Y}(\lambda) = (I + O(\lambda)) \lambda^\Lambda C = T(I + O(\lambda)) \lambda^{(\sigma/2)J} T^{-1} C,$$

where  $T$  is as in Lemma 4.1. Using the connection formula

$$e^{z/2} z^{-1/2} W_{\kappa, \sigma/2}(z) = \frac{\Gamma(-\sigma) z^{\sigma/2}}{\Gamma((1 - \sigma)/2 - \kappa)} (1 + O(z)) + \frac{\Gamma(\sigma) z^{-\sigma/2}}{\Gamma((1 + \sigma)/2 - \kappa)} (1 + O(z))$$

(cf. [1, 13.1.3, 13.1.33]), we compare the behaviours around  $\lambda = 0$  to derive  $C$  as in (1).

If  $\sigma = 0$  and  $\theta_\infty \neq 0$ , then, around  $\lambda = 0$ ,

$$\hat{Y}(\lambda) = (I + O(\lambda)) \lambda^\Lambda C = T(I + O(\lambda)) \lambda^\Delta T^{-1} C,$$

where  $T$  and  $\Lambda$  are as in Lemma 4.2. Note that (1,1)- and (1,2)-entries of  $T(I + O(\lambda)) \lambda^\Delta$  are  $-\theta_\infty/2 + O(\lambda)$  and  $-(\theta_\infty/2) \log \lambda + 1 + O(\lambda)$ , respectively. On the other hand, using

$$e^{z/2} z^{-1/2} W_{\kappa, 0}(z) = -\frac{1}{\Gamma(1/2 - \kappa)} \left( (1 + O(z)) \log z + \psi(1/2 - \kappa) - 2\psi(1) + O(z) \right)$$

(cf. [1, 13.1.6]), we can also see how  $\hat{Y}(\lambda)$  behaves around  $\lambda = 0$ . Comparison of these leads us to  $C$  as in (i) of (2). Under the supposition  $\sigma = \theta_\infty = 0$ , if  $\Lambda = \Delta$  (respectively,  $\Lambda = \Delta_-$ ), then (8.1) admits the matrix solution

$$\begin{pmatrix} e^{\lambda/2} & -\lambda^{-1/2} W_{-1/2, 0}(\lambda) \\ 0 & e^{-\lambda/2} \end{pmatrix} \quad \left( \text{respectively, } \begin{pmatrix} e^{\lambda/2} & 0 \\ e^{-\pi i/2} \lambda^{-1/2} W_{-1/2, 0}(e^{-\pi i} \lambda) & e^{-\lambda/2} \end{pmatrix} \right).$$

Using the connection formula above together with  $T$  in each case, we find the matrices in (ii) and (iii).  $\square$

**9.2. Hypergeometric system.** Let us begin with

$$\frac{d\mathbf{u}}{dz} = \left( \frac{1}{z} \begin{pmatrix} 0 & 1 \\ 0 & 1 - \gamma \end{pmatrix} + \frac{1}{z-1} \begin{pmatrix} 0 & 0 \\ -\alpha\beta & \gamma - \alpha - \beta - 1 \end{pmatrix} \right) \mathbf{u}, \quad \mathbf{u} = \begin{pmatrix} u \\ zu' \end{pmatrix},$$

in which  $u$  solves the hypergeometric equation  $z(1-z)u'' + (\gamma - (\alpha + \beta + 1)z)u' - \alpha\beta u = 0$ . The eigenvalues of the residue matrices are  $0, 1 - \gamma$  at  $z = 0$ ,  $0, \gamma - \alpha - \beta - 1$  at  $z = 1$  and  $\alpha, \beta$  at  $z = \infty$ . Under the supposition  $\alpha - \beta \notin \mathbb{Z}$ , diagonalising the residue matrix at  $z = \infty$  and shifting the eigenvalues to  $\pm(1 - \gamma)/2$ ,  $\pm(\gamma - \alpha - \beta - 1)/2$ ,  $\pm(\beta - \alpha)/2$ , we obtain the system

$$(9.1) \quad \frac{d\Psi}{dz} = \left( \frac{B_0}{z} + \frac{B_1}{z-1} \right) \Psi,$$

$$B_0 = \frac{1}{\alpha - \beta} \begin{pmatrix} (1 - \gamma)(\alpha + \beta)/2 + \alpha\beta & \beta(\beta - \gamma + 1)/R \\ \alpha(\gamma - \alpha - 1)R & -(1 - \gamma)(\alpha + \beta)/2 - \alpha\beta \end{pmatrix},$$

$$B_1 = \frac{1}{\alpha - \beta} \begin{pmatrix} (\gamma - \alpha - \beta - 1)(\alpha + \beta)/2 + \alpha\beta & \beta(\gamma - \beta - 1)/R \\ \alpha(\alpha - \gamma + 1)R & -(\gamma - \alpha - \beta - 1)(\alpha + \beta)/2 - \alpha\beta \end{pmatrix}$$

with given  $R \neq 0$ , which has the following property.

**Proposition 9.2.** *Suppose that  $\alpha - \beta \notin \mathbb{Z}$ . System (9.1) admits the matrix solution*

$$\Psi(z) = z^{-(1-\gamma)/2} (z-1)^{-(\gamma-\alpha-\beta-1)/2} \text{diag}[1, R] [\mathbf{v}_\alpha \ \mathbf{v}_\beta] \text{diag}[1, 1/R]$$

with

$$[\mathbf{v}_\alpha \ \mathbf{v}_\beta] = P^{-1} \begin{pmatrix} u_\alpha & u_\beta \\ zu'_\alpha & zu'_\beta \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ -\alpha & -\beta \end{pmatrix},$$

$$u_\alpha := z^{-\alpha} F(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, z^{-1}),$$

$$u_\beta := z^{-\beta} F(\beta, \beta - \gamma + 1, \beta - \alpha + 1, z^{-1}),$$

which satisfies

$$\Psi(z) = (I + O(z^{-1})) z^{(\beta-\alpha)J/2} \quad z \rightarrow \infty.$$

We substitute the connection formulas representing  $u_\alpha$  and  $u_\beta$  as linear combinations of hypergeometric series around  $z = 0$  or  $z = 1$  (cf. [3, §2.9]) and choose gauge matrices suitably to obtain

**Proposition 9.3.** *Suppose that  $\alpha - \beta, \gamma, \gamma - \alpha - \beta \notin \mathbb{Z}$ . Then*

$$\begin{aligned} \Psi(z)|_{R=1} &= G_0(I + O(z)) z^{(1-\gamma)J/2} \tilde{C}_0 & z \rightarrow 0, \\ &= G_1(I + O(z-1)) (z-1)^{(\gamma-\alpha-\beta-1)J/2} \tilde{C}_1 & z \rightarrow 1 \end{aligned}$$

for  $|\arg z - \pi| < \pi$ ,  $|\arg(z-1) - \pi| < \pi$ , where

$$\tilde{C}_0 = \begin{pmatrix} \frac{e^{\pi i(\gamma-\alpha-1)} \Gamma(1-\beta+\alpha) \Gamma(\gamma-1)}{\Gamma(\alpha) \Gamma(\gamma-\beta)} & \frac{e^{\pi i(\gamma-\beta-1)} \Gamma(1-\alpha+\beta) \Gamma(\gamma-1)}{\Gamma(\beta) \Gamma(\gamma-\alpha)} \\ \frac{e^{-\pi i \alpha} \Gamma(1-\beta+\alpha) \Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha) \Gamma(1-\beta)} & \frac{e^{-\pi i \beta} \Gamma(1-\alpha+\beta) \Gamma(1-\gamma)}{\Gamma(1-\gamma+\beta) \Gamma(1-\alpha)} \end{pmatrix},$$

$$\tilde{C}_1 = \begin{pmatrix} \frac{\Gamma(1-\beta+\alpha)\Gamma(\alpha+\beta-\gamma)}{\Gamma(1-\gamma+\alpha)\Gamma(\alpha)} & \frac{\Gamma(1-\alpha+\beta)\Gamma(\alpha+\beta-\gamma)}{\Gamma(1-\gamma+\beta)\Gamma(\beta)} \\ \frac{\Gamma(1-\beta+\alpha)\Gamma(\gamma-\alpha-\beta)}{\Gamma(1-\beta)\Gamma(\gamma-\beta)} & \frac{\Gamma(1-\alpha+\beta)\Gamma(\gamma-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(\gamma-\alpha)} \end{pmatrix}.$$

In the case where  $\alpha = \beta$ , we transform the system in such a way that the residue matrix at  $z = \infty$  becomes  $\Delta$ . The result of this procedure is

$$(9.2) \quad \frac{d\Psi}{dz} = \left( \frac{B_0}{z} + \frac{B_1}{z-1} \right) \Psi,$$

$$B_0 = \begin{pmatrix} (\gamma-2\alpha-1)/2 & 1 \\ \alpha(\gamma-\alpha-1) & -(\gamma-2\alpha-1)/2 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} (2\alpha-\gamma+1)/2 & 0 \\ \alpha(\alpha-\gamma+1) & -(2\alpha-\gamma+1)/2 \end{pmatrix}.$$

Write, for  $m \in \mathbb{N}$ ,

$$F_{\log}(\alpha, \beta, m, z) := F(\alpha, \beta, m, z) \log z + \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n}{(m)_n n!} \psi_n(\alpha, \beta, m) z^n - \sum_{n=1}^{m-1} \frac{(n-1)!(1-m)_n}{(1-\alpha)_n (1-\beta)_n} z^{-n}$$

with

$$\begin{aligned} \psi_n(\alpha, \beta, m) &= \psi(\alpha+n) - \psi(\alpha) + \psi(\beta+n) - \psi(\beta) \\ &\quad - \psi(m+n) + \psi(m) - \psi(1+n) + \psi(1). \end{aligned}$$

Then  $z^{-\alpha} F_{\log}(\alpha, \alpha-\gamma+1, 1, z^{-1})$  is a logarithmic hypergeometric function near  $z = \infty$ , which satisfies, for  $|\arg z - \pi| < \pi$ ,

$$(9.3) \quad \begin{aligned} -z^{-\alpha} F_{\log}(\alpha, \alpha-\gamma+1, 1, z^{-1}) &= \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} e^{-\pi i \alpha} F(\alpha, \alpha, \gamma, z) \\ &\quad - (2\psi(1) - \psi(\alpha) - \psi(\gamma-\alpha) - \pi i) z^{-\alpha} F(\alpha, \alpha-\gamma+1, 1, z^{-1}) \end{aligned}$$

(cf. [1, 15.3.13], [3, §2.10]). Then we have

**Proposition 9.4.** *System (9.2) admits the matrix solution*

$$\Psi(z) = z^{-(1-\gamma)/2} (z-1)^{-(\gamma-2\alpha-1)/2} [\mathbf{v}_{\alpha} \ \mathbf{v}_{\alpha \log}]$$

with

$$\begin{aligned} [\mathbf{v}_{\alpha} \ \mathbf{v}_{\alpha \log}] &= P^{-1} \begin{pmatrix} u_{\alpha} & u_{\alpha \log} \\ zu'_{\alpha} & zu'_{\alpha \log} \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix}, \\ u_{\alpha} &:= z^{-\alpha} F(\alpha, \alpha-\gamma+1, 1, z^{-1}), \\ u_{\alpha \log} &:= -z^{-\alpha} F_{\log}(\alpha, \alpha-\gamma+1, 1, z^{-1}), \end{aligned}$$

and  $\Psi(z)$  satisfies

$$\Psi(z) = (I + O(z^{-1})) z^{\Delta} \quad z \rightarrow \infty.$$

In [3, §2.10, (3)] replacing  $z$  by  $1 - z$  and applying limiting procedure, we get, for  $|\arg z| < \pi$ ,

$$(9.4) \quad F(\alpha, \alpha, 2\alpha - \gamma + 1, 1 - z) = \frac{\Gamma(2\alpha - \gamma + 1)}{\Gamma(\alpha)\Gamma(\alpha - \gamma + 1)} z^{-\alpha} \left( -F_{\log}(\alpha, \alpha - \gamma + 1, 1, z^{-1}) \right. \\ \left. - (\psi(\alpha) + \psi(\alpha - \gamma + 1) - 2\psi(1))F(\alpha, \alpha - \gamma + 1, 1, z^{-1}) \right).$$

Using (9.3) and this relation, we have the following:

(i) if  $\gamma \notin \mathbb{Z}$ , then, for  $|\arg z - \pi| < \pi$ ,

$$z^{-\alpha} F(\alpha, \alpha - \gamma + 1, 1, z^{-1}) = \frac{e^{-\pi i \alpha} \Gamma(1 - \gamma)}{\Gamma(1 - \alpha) \Gamma(\alpha - \gamma + 1)} f_0(z) - \frac{e^{\pi i (\gamma - \alpha)} \Gamma(\gamma - 1)}{\Gamma(\alpha) \Gamma(\gamma - \alpha)} g_0(z), \\ -z^{-\alpha} F_{\log}(\alpha, \alpha - \gamma + 1, 1, z^{-1}) \\ = -\frac{e^{-\pi i \alpha} (2\psi(1) - \psi(1 - \alpha) - \psi(\alpha - \gamma + 1) - \pi i) \Gamma(1 - \gamma)}{\Gamma(1 - \alpha) \Gamma(\alpha - \gamma + 1)} f_0(z) \\ + \frac{e^{\pi i (\gamma - \alpha)} (2\psi(1) - \psi(\alpha) - \psi(\gamma - \alpha) - \pi i) \Gamma(\gamma - 1)}{\Gamma(\alpha) \Gamma(\gamma - \alpha)} g_0(z)$$

with

$$f_0(z) = F(\alpha, \alpha, \gamma, z), \quad g_0(z) = z^{1-\gamma} F(\alpha - \gamma + 1, \alpha - \gamma + 1, 2 - \gamma, z);$$

(ii) if  $\gamma - 2\alpha \notin \mathbb{Z}$ , then, for  $|\arg z| < \pi$ ,  $|\arg(1 - z)| < \pi$ ,

$$z^{-\alpha} F(\alpha, \alpha - \gamma + 1, 1, z^{-1}) = \frac{\Gamma(\gamma - 2\alpha)}{\Gamma(1 - \alpha) \Gamma(\gamma - \alpha)} f_1(z) + \frac{e^{\pi i (\gamma - 2\alpha)} \Gamma(2\alpha - \gamma)}{\Gamma(\alpha) \Gamma(\alpha - \gamma + 1)} g_1(z), \\ -z^{-\alpha} F_{\log}(\alpha, \alpha - \gamma + 1, 1, z^{-1}) \\ = \frac{(\psi(1 - \alpha) + \psi(\gamma - \alpha) - 2\psi(1)) \Gamma(\gamma - 2\alpha)}{\Gamma(1 - \alpha) \Gamma(\gamma - \alpha)} f_1(z) \\ + \frac{e^{\pi i (\gamma - 2\alpha)} (\psi(\alpha) + \psi(\alpha - \gamma + 1) - 2\psi(1)) \Gamma(2\alpha - \gamma)}{\Gamma(\alpha) \Gamma(\alpha - \gamma + 1)} g_1(z)$$

with

$$f_1(z) = F(\alpha, \alpha, 2\alpha - \gamma + 1, 1 - z), \\ g_1(z) = (1 - z)^{\gamma - 2\alpha} F(\gamma - \alpha, \gamma - \alpha, \gamma - 2\alpha + 1, 1 - z).$$

Then we have

**Proposition 9.5.** *Suppose that  $\alpha = \beta$  and that  $\gamma, \gamma - 2\alpha \notin \mathbb{Z}$ . Then*

$$\Psi(z) = G_0(I + O(z)) z^{(1-\gamma)J/2} \tilde{C}_0 \quad z \rightarrow 0, \\ = G_1(I + O(z - 1)) (z - 1)^{(\gamma - 2\alpha - 1)J/2} \tilde{C}_1 \quad z \rightarrow 1$$

for  $0 < \arg z < \pi$ ,  $|\arg(z - 1) - \pi| < \pi$ , where

$$\tilde{C}_0 = \begin{pmatrix} -\frac{e^{\pi i (\gamma - \alpha)} \Gamma(\gamma - 1)}{\Gamma(\alpha) \Gamma(\gamma - \alpha)} & \frac{e^{\pi i (\gamma - \alpha)} \hat{\psi}_{12}^0(\alpha, \gamma) \Gamma(\gamma - 1)}{\Gamma(\alpha) \Gamma(\gamma - \alpha)} \\ \frac{e^{-\pi i \alpha} \Gamma(1 - \gamma)}{\Gamma(1 - \alpha) \Gamma(\alpha - \gamma + 1)} & -\frac{e^{-\pi i \alpha} \hat{\psi}_{22}^0(\alpha, \gamma) \Gamma(1 - \gamma)}{\Gamma(1 - \alpha) \Gamma(\alpha - \gamma + 1)} \end{pmatrix},$$

$$\tilde{C}_1 = \begin{pmatrix} \frac{\Gamma(2\alpha - \gamma)}{\Gamma(\alpha)\Gamma(\alpha - \gamma + 1)} & \frac{\hat{\psi}_{12}^1(\alpha, \gamma)\Gamma(2\alpha - \gamma)}{\Gamma(\alpha)\Gamma(\alpha - \gamma + 1)} \\ \frac{\Gamma(\gamma - 2\alpha)}{\Gamma(1 - \alpha)\Gamma(\gamma - \alpha)} & \frac{\hat{\psi}_{22}^1(\alpha, \gamma)\Gamma(\gamma - 2\alpha)}{\Gamma(1 - \alpha)\Gamma(\gamma - \alpha)} \end{pmatrix}$$

with

$$\begin{aligned} \hat{\psi}_{12}^0(\alpha, \gamma) &= 2\psi(1) - \psi(\alpha) - \psi(\gamma - \alpha) - \pi i, \\ \hat{\psi}_{22}^0(\alpha, \gamma) &= 2\psi(1) - \psi(1 - \alpha) - \psi(\alpha - \gamma + 1) - \pi i, \\ \hat{\psi}_{12}^1(\alpha, \gamma) &= \psi(\alpha) + \psi(\alpha - \gamma + 1) - 2\psi(1), \\ \hat{\psi}_{22}^1(\alpha, \gamma) &= \psi(1 - \alpha) + \psi(\gamma - \alpha) - 2\psi(1). \end{aligned}$$

**9.3. Non-generic cases.** Suppose that  $\alpha - \beta \notin \mathbb{Z}$ . In the case where  $\gamma \in \mathbb{Z}$ , the hypergeometric function behaves logarithmically around  $z = 0$ . Under the condition  $\alpha, \beta \notin \mathbb{Z}$ , for  $|\arg z - \pi| < \pi$ , application of limiting procedure to connection formulas in the generic case yields the following (cf. [3, §2.10], [1, 15.5.17, 15.5.19, 15.5.21]):

(i) if  $1 - \gamma = l = 0, 1, 2, \dots$ ,

$$\begin{aligned} & z^{-\alpha} F(\alpha, \alpha + l, \alpha - \beta + 1, z^{-1}) \\ &= a_-(\alpha, \beta, l) \left( z^l F_{\log}(\alpha + l, \beta + l, 1 + l, z) + b_-(\alpha, \beta, l) z^l F(\alpha + l, \beta + l, 1 + l, z) \right), \end{aligned}$$

where

$$\begin{aligned} a_-(\alpha, \beta, l) &= \frac{(-1)^{l-1} e^{-\pi i \alpha} \Gamma(\alpha - \beta + 1) \Gamma(\beta + l)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(1 - \beta) l!}, \\ b_-(\alpha, \beta, l) &= \psi(\alpha + l) + \psi(\beta + l) - \psi(1 + l) - \psi(1) - \psi(\beta) + \psi(1 - \beta) - \pi i; \end{aligned}$$

(ii) if  $1 - \gamma = l = -1, -2, -3, \dots$ ,

$$\begin{aligned} & z^{-\alpha} F(\alpha, \alpha + l, \alpha - \beta + 1, z^{-1}) \\ &= a_+(\alpha, \beta, l) \left( F_{\log}(\alpha, \beta, 1 - l, z) + b_+(\alpha, \beta, l) F(\alpha, \beta, 1 - l, z) \right), \end{aligned}$$

where

$$\begin{aligned} a_+(\alpha, \beta, l) &= -\frac{e^{-\pi i \alpha} \Gamma(\alpha - \beta + 1) \Gamma(\beta)}{\Gamma(\alpha + l) \Gamma(\beta + l) \Gamma(1 - \beta - l) (-l)!}, \\ b_+(\alpha, \beta, l) &= \psi(\alpha) + \psi(\beta) - \psi(1 - l) - \psi(1) - \psi(\beta + l) + \psi(1 - \beta - l) - \pi i. \end{aligned}$$

In these cases,  $\Psi(z)|_{R=1}$  for (9.1) around  $z = 0$  has the form  $G_0(I + O(z))E(z)\tilde{C}_0$ , where  $E(z)$  is  $z^{lJ/2}z^\Delta$  if  $1 - \gamma = l = 0, 1, 2, \dots$ , and  $z^{lJ/2}z^{\Delta-}$  if  $1 - \gamma = l = -1, -2, -3, \dots$ , and hence we have

**Proposition 9.6.** *Suppose that  $\alpha, \beta, \alpha - \beta \notin \mathbb{Z}$  and  $1 - \gamma = l \in \mathbb{Z}$ . For  $|\arg z - \pi| < \pi$ ,*

$$\tilde{C}_0 = \begin{pmatrix} b_-(\alpha, \beta, l) & b_-(\beta, \alpha, l) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_-(\alpha, \beta, l) & 0 \\ 0 & a_-(\beta, \alpha, l) \end{pmatrix} \quad \text{if } l = 0, 1, 2, \dots,$$

$$= \begin{pmatrix} 1 & 1 \\ b_+(\alpha, \beta, l) & b_+(\beta, \alpha, l) \end{pmatrix} \begin{pmatrix} a_+(\alpha, \beta, l) & 0 \\ 0 & a_+(\beta, \alpha, l) \end{pmatrix} \quad \text{if } l = -1, -2, -3, \dots$$

From  $F(\alpha, \beta, \gamma, z) = (1 - z)^{-\alpha} F(\alpha, \gamma - \beta, \gamma, z/(z - 1))$  [1, 15.3.4], it follows that

$$z^{-\alpha} F(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, z^{-1}) = e^{\pi i \alpha} \zeta^{-\alpha} F(\alpha, \gamma - \beta, \alpha - \beta + 1, \zeta^{-1})$$

for  $|\arg(1 - z) - \pi| < \pi$ ,  $|\arg z - \pi| < \pi$ , where  $\zeta = 1 - z = e^{\pi i}(z - 1)$ . In the case where  $\gamma - \alpha - \beta \in \mathbb{Z}$ , using this relation, from the connection formulas above we immediately obtain the following:

(i) if  $\gamma - \alpha - \beta = l = 0, 1, 2, \dots$ ,

$$\begin{aligned} & z^{-\alpha} F(\alpha, 1 - l - \beta, \alpha - \beta + 1, z^{-1}) \\ &= e^{\pi i \alpha} a_-(\alpha, \beta, l) \left( (1 - z)^l F_{\log}(\alpha + l, \beta + l, 1 + l, 1 - z) \right. \\ & \quad \left. + b_-(\alpha, \beta, l) (1 - z)^l F(\alpha + l, \beta + l, 1 + l, 1 - z) \right); \end{aligned}$$

(ii) if  $\gamma - \alpha - \beta = l = -1, -2, -3, \dots$ ,

$$\begin{aligned} & z^{-\alpha} F(\alpha, 1 - l - \beta, \alpha - \beta + 1, z^{-1}) \\ &= e^{\pi i \alpha} a_+(\alpha, \beta, l) (F_{\log}(\alpha, \beta, 1 - l, 1 - z) + b_+(\alpha, \beta, l) F(\alpha, \beta, 1 - l, 1 - z)). \end{aligned}$$

Observing that  $(1 - z)^{lJ/2} (1 - z)^\Delta = (-1)^l (I + O(z - 1)) (z - 1)^{lJ/2} (z - 1)^\Delta$ , we have

**Proposition 9.7.** *Suppose that  $\alpha, \beta, \alpha - \beta \notin \mathbb{Z}$  and  $\gamma - \alpha - \beta = l \in \mathbb{Z}$ . For  $|\arg z - \pi| < \pi$ ,  $|\arg(z - 1)| < \pi$ ,*

$$\begin{aligned} \tilde{C}_1 &= e^{\pi i \alpha} \begin{pmatrix} b_-(\alpha, \beta, l) & b_-(\beta, \alpha, l) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_-(\alpha, \beta, l) & 0 \\ 0 & a_-(\beta, \alpha, l) \end{pmatrix} \quad \text{if } l = 0, 1, 2, \dots, \\ &= e^{\pi i \alpha} \begin{pmatrix} 1 & 1 \\ b_+(\alpha, \beta, l) & b_+(\beta, \alpha, l) \end{pmatrix} \begin{pmatrix} a_+(\alpha, \beta, l) & 0 \\ 0 & a_+(\beta, \alpha, l) \end{pmatrix} \quad \text{if } l = -1, -2, -3, \dots \end{aligned}$$

For  $\alpha \notin \mathbb{Z}$ , putting  $\alpha = \beta$ , we have the following:

(i) if  $1 - \gamma = l = 0, 1, 2, \dots$ ,

$$\begin{aligned} & z^{-\alpha} F(\alpha, \alpha + l, 1, z^{-1}) \\ &= a_-(\alpha, \alpha, l) (z^l F_{\log}(\alpha + l, \alpha + l, 1 + l, z) + b_-(\alpha, \alpha, l) z^l F(\alpha + l, \alpha + l, 1 + l, z)); \end{aligned}$$

(ii) if  $1 - \gamma = l = -1, -2, -3, \dots$ ,

$$\begin{aligned} & z^{-\alpha} F(\alpha, \alpha + l, 1, z^{-1}) \\ &= a_+(\alpha, \alpha, l) (F_{\log}(\alpha, \alpha, 1 - l, z) + b_+(\alpha, \alpha, l) F(\alpha, \alpha, 1 - l, z)). \end{aligned}$$

Combining these formulas with (9.3), we have another relation in the case where  $\alpha = \beta \notin \mathbb{Z}$ ,  $\gamma \in \mathbb{Z}$ . For example, if  $1 - \gamma = l = 0, 1, 2, \dots$ , observing that

$$\frac{1}{\Gamma(\gamma)} F(\alpha, \alpha, \gamma, z) \Big|_{\gamma=-l+1} = \frac{\Gamma(\alpha + l)^2}{\Gamma(\alpha)^2 l!} z^l F(\alpha + l, \alpha + l, 1 + l, z),$$



we get

$$\begin{aligned} -z^{-\alpha}F_{\log}(\alpha, \alpha+l, 1, z^{-1}) &= \frac{\Gamma(\alpha+l)^2\Gamma(1-\alpha-l)}{\Gamma(\alpha)l!}e^{-\pi i\alpha}z^lF(\alpha+l, \alpha+l, 1+l, z) \\ &\quad - (2\psi(1) - \psi(\alpha) - \psi(1-\alpha-l) - \pi i)a_-(\alpha, \alpha, l) \\ &\quad \times \left( z^lF_{\log}(\alpha+l, \alpha+l, 1+l, z) + b_-(\alpha, \alpha, l)z^lF(\alpha+l, \alpha+l, 1+l, z) \right) \end{aligned}$$

for  $|\arg z - \pi| < \pi$ . On the other hand, in the case where  $\gamma - 2\alpha \in \mathbb{Z}$ , we have the relations:

(i) if  $\gamma - 2\alpha = l = 0, 1, 2, \dots$ ,

$$\begin{aligned} &z^{-\alpha}F(\alpha, 1-l-\alpha, 1, z^{-1}) \\ &= e^{\pi i\alpha}a_-(\alpha, \alpha, l) \left( (1-z)^lF_{\log}(\alpha+l, \alpha+l, 1+l, 1-z) \right. \\ &\quad \left. + b_-(\alpha, \alpha, l)(1-z)^lF(\alpha+l, \alpha+l, 1+l, 1-z) \right); \end{aligned}$$

(ii) if  $\gamma - 2\alpha = l = -1, -2, -3, \dots$ ,

$$\begin{aligned} &z^{-\alpha}F(\alpha, 1-l-\alpha, 1, z^{-1}) \\ &= e^{\pi i\alpha}a_+(\alpha, \alpha, l) (F_{\log}(\alpha, \alpha, 1-l, 1-z) + b_+(\alpha, \alpha, l)F(\alpha, \alpha, 1-l, 1-z)). \end{aligned}$$

The connection formula for  $-z^{-\alpha}F_{\log}(\alpha, 1-l-\alpha, 1, z^{-1})$  is obtained by an analogous argument, in which we use (9.4). Thus we have

**Proposition 9.8.** (1) Suppose that  $\alpha = \beta \notin \mathbb{Z}$  and  $1-\gamma = l \in \mathbb{Z}$ . For  $|\arg z - \pi| < \pi$ ,

$$\begin{aligned} \tilde{C}_0 &= \begin{pmatrix} b_-(\alpha, \alpha, l) & b_-(\alpha, \alpha, l) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_-(\alpha, \alpha, l) & 0 \\ 0 & \psi_0(\alpha, l)a_-(\alpha, \alpha, l) \end{pmatrix} \\ &\quad + \frac{\Gamma(\alpha+l)^2\Gamma(1-\alpha-l)}{\Gamma(\alpha)l!}e^{-\pi i\alpha}\Delta \quad \text{if } l = 0, 1, 2, \dots, \\ &= \begin{pmatrix} 1 & 1 \\ b_+(\alpha, \alpha, l) & b_+(\alpha, \alpha, l) \end{pmatrix} \begin{pmatrix} a_+(\alpha, \alpha, l) & 0 \\ 0 & \psi_0(\alpha, l)a_+(\alpha, \alpha, l) \end{pmatrix} \\ &\quad + \frac{\Gamma(\alpha)\Gamma(1-\alpha-l)}{2\Gamma(1-l)}e^{-\pi i\alpha}(I-J) \quad \text{if } l = -1, -2, -3, \dots, \end{aligned}$$

where  $\psi_0(\alpha, l) = \psi(\alpha) + \psi(1-\alpha-l) - 2\psi(1) + \pi i$ .

(2) Suppose that  $\alpha = \beta \notin \mathbb{Z}$  and  $\gamma - 2\alpha = l \in \mathbb{Z}$ . For  $0 < \arg z < \pi$ ,  $|\arg(z-1)| < \pi$ ,

$$\begin{aligned} \tilde{C}_1 &= e^{\pi i\alpha} \begin{pmatrix} b_-(\alpha, \alpha, l) & b_-(\alpha, \alpha, l) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_-(\alpha, \alpha, l) & 0 \\ 0 & \psi_1(\alpha, l)a_-(\alpha, \alpha, l) \end{pmatrix} \\ &\quad + \frac{\Gamma(\alpha+l)^2\Gamma(1-\alpha-l)}{\Gamma(\alpha)l!}\Delta \quad \text{if } l = 0, 1, 2, \dots, \\ &= e^{\pi i\alpha} \begin{pmatrix} 1 & 1 \\ b_+(\alpha, \alpha, l) & b_+(\alpha, \alpha, l) \end{pmatrix} \begin{pmatrix} a_+(\alpha, \alpha, l) & 0 \\ 0 & \psi_1(\alpha, l)a_+(\alpha, \alpha, l) \end{pmatrix} \end{aligned}$$

$$+ \frac{\Gamma(\alpha)\Gamma(1-\alpha-l)}{2\Gamma(1-l)}(I-J) \quad \text{if } l = -1, -2, -3, \dots,$$

where  $\psi_1(\alpha, l) = \psi(\alpha) + \psi(1-\alpha-l) - 2\psi(1)$ .

## 10. PROOFS OF THEOREMS 2.13 THROUGH 2.16

**10.1. Proofs of Theorems 2.13 and 2.14.** In (9.1) and Proposition 9.2 set  $(1-\gamma)/2 = \theta_0/2$ ,  $(\gamma - \alpha - \beta - 1)/2 = \theta_x/2$ ,  $(\beta - \alpha)/2 = \sigma/2$ , that is,  $\alpha = -(\sigma + \theta_0 + \theta_x)/2$ ,  $\beta = (\sigma - \theta_0 - \theta_x)/2$ ,  $\gamma = 1 - \theta_0$ , and  $R = \beta(\beta - \gamma + 1)/(\alpha - \beta) = (\theta_0^2 - (\sigma - \theta_x)^2)/(4\sigma)$ . Then we have  $TB_0T^{-1} = \Lambda_0$ ,  $TB_1T^{-1} = \Lambda_x$ ,  $T(\sigma J/2)T^{-1} = \Lambda$ , where  $\Lambda_0$ ,  $\Lambda_x$ ,  $T$ ,  $\Lambda$  are as in Lemma 4.1, and  $Z = T\Psi(\lambda)$  satisfies

$$\frac{dZ}{d\lambda} = \left( \frac{\Lambda_0}{\lambda} + \frac{\Lambda_x}{\lambda - 1} \right) Z.$$

Furthermore, by

$$Z = \tilde{\rho}^{-\Lambda} Y = (\rho_*^{1/\sigma})^{-T(\sigma J/2)T^{-1}} Y = T\rho_*^{-J/2} T^{-1} Y$$

this is changed into a system of the form (8.2)

$$\frac{dY}{d\lambda} = \left( \frac{\tilde{\Lambda}_0}{\lambda} + \frac{\tilde{\Lambda}_x}{\lambda - 1} \right) Y,$$

which has a matrix solution such that

$$\begin{aligned} Y &= T\rho_*^{J/2} T^{-1} Z = T\rho_*^{J/2} \Psi(\lambda) = T\rho_*^{J/2} (I + O(\lambda^{-1})) \lambda^{\sigma J/2} \\ &= T(I + O(\lambda^{-1})) \lambda^{\sigma J/2} \rho_*^{J/2} = (I + O(\lambda^{-1})) \lambda^{\Lambda} T\rho_*^{J/2} \end{aligned}$$

near  $\lambda = \infty$ . Hence the connection matrices  $C^{(0)}$ ,  $C^{(x)}$  are those for

$$\tilde{Y} = Y(T\rho_*^{J/2})^{-1} = T\rho_*^{J/2} \Psi(\lambda) (T\rho_*^{J/2})^{-1} = (I + O(\lambda^{-1})) \lambda^{\Lambda}$$

corresponding to (8.3). By Proposition 9.3, the connection matrices for  $\Psi(\lambda)$  are  $\tilde{C}_{\iota/x} \text{diag}[1, 1/R]$  ( $\iota = 0, x$ ) under the supposition  $\gamma, \gamma - \alpha - \beta \notin \mathbb{Z}$ , that is,  $\theta_0, \theta_x \notin \mathbb{Z}$ , and we may choose  $C^{(\iota)} = C_{\iota} T^{-1}$  in such a way that

$$(10.1) \quad C_{\iota} = \rho_*^{I/2} \tilde{C}_{\iota/x} \text{diag}[1, 1/R] \rho_*^{-J/2} \quad (\iota = 0, x)$$

with  $\alpha, \beta, \gamma, R$  set as above (cf. Remark 8.1). Combining  $C$  in Proposition 9.1 with  $C^{(0)}$ ,  $C^{(x)}$ , from (8.4) we obtain  $M_0, M_x$  in Theorem 2.13. Theorem 2.14 is shown by limiting procedure  $\sigma \rightarrow \sigma_0$ . In (10.1), replacing  $\rho_*$  by  $\rho$  and letting  $\sigma \rightarrow \theta_0 \pm \theta_x, \theta_x - \theta_0$ , we obtain  $C_0^*$  and  $C_1^*$  as in Theorem 2.14 other than  $C_0^*$  for  $\sigma_0 = \theta_x - \theta_0$ . By Remark 8.1, we may choose  $\text{diag}[(\sigma + \theta_0 - \theta_x)/2, ((\sigma + \theta_0 - \theta_x)/2)^{-1}] \tilde{C}_0$  instead of  $\tilde{C}_0$ . In (10.1) with such a matrix, letting  $\sigma \rightarrow \theta_x - \theta_0$  we derive  $C_0^*$  with  $\tilde{C}_0^*$  in (iii).

**10.2. Proof of Theorem 2.15.** Suppose that  $\theta_\infty \neq 0$ . In (9.2) and Proposition 9.4 set  $1 - \gamma = \theta_0$ ,  $\gamma - 2\alpha - 1 = \mp\theta_x$ , that is,  $\alpha = -(\theta_0 \mp \theta_x)/2$ ,  $\gamma = 1 - \theta_0$ . Then, for  $\Lambda_0$ ,  $\Lambda_x$ ,  $T$ ,  $\Lambda$  as in Lemma 4.2, we have  $TB_0T^{-1} = \Lambda_0$ ,  $TB_1T^{-1} = \Lambda_x$ ,  $T\Delta T^{-1} = \Lambda$ , and, near  $\lambda = \infty$ ,

$$\begin{aligned} Y &= T\rho^\Delta\Psi(\lambda) = T\rho^\Delta(I + O(\lambda^{-1}))\lambda^\Delta \\ &= T(I + O(\lambda^{-1}))\lambda^\Delta\rho^\Delta = (I + O(\lambda^{-1}))\lambda^\Delta T\rho^\Delta \end{aligned}$$

solves system (8.2). Hence the connection matrices  $C^{(0)\pm}$ ,  $C^{(x)\pm}$  for

$$\tilde{Y} = Y(T\rho^\Delta)^{-1} = T\rho^\Delta\Psi(\lambda)(T\rho^\Delta)^{-1} = (I + O(\lambda^{-1}))\lambda^\Delta$$

are desired ones, which are written as  $C_\iota^\pm T^{-1}$  with  $C_\iota^\pm = \tilde{C}_{\iota/x}^\pm \rho^{-\Delta}$  ( $\iota = 0, x$ ), where  $\tilde{C}_{\iota/x}^\pm$  are given by Proposition 9.5. For  $y_{\text{ilog}}(\rho, x)$  and  $y_{\text{ilog}}^{(l)}(\rho, x)$  with  $\theta_0^2 - \theta_x^2 \neq 0$ , we restrict to  $\alpha = -(\theta_0 + \theta_x)/2$  and replace  $\rho$  by  $\rho \exp(-2\theta_x(\theta_0^2 - \theta_x^2)^{-1})$  according to Proposition 5.1. In the case where  $\alpha = -(\theta_0 - \theta_x)/2$ ,  $\tilde{C}_1^+$  should be replaced by  $K^+\tilde{C}_1^+$ , because the local solution of (9.2) around  $\lambda = 1$  has the form  $G_1(I + O(\lambda - 1))(\lambda - 1)^{-(\theta_x/2)J}$ . If  $\theta_\infty = 0$ , in the argument above the matrix  $T$  is to be replaced by those in (2) of Lemma 4.2. Thus we obtain the monodromy matrices in Theorem 2.15.

**10.3. Proof of Theorem 2.16.** The systems corresponding to (8.1) and (8.2) are

$$\frac{d\hat{Y}}{d\lambda} = \frac{J}{2}\hat{Y}, \quad \frac{d\tilde{Y}}{d\lambda} = \left(\frac{\Lambda_0}{\lambda} - \frac{\Lambda_0}{\lambda - 1}\right)\tilde{Y}$$

having the matrix solutions  $\hat{Y} = \exp((J/2)\lambda)$  and

$$\begin{aligned} \tilde{Y} &= T \begin{pmatrix} \lambda^{\theta_0/2}(\lambda - 1)^{-\theta_0/2} & 0 \\ 0 & \lambda^{-\theta_0/2}(\lambda - 1)^{\theta_0/2} \end{pmatrix} T^{-1} & \text{if } \theta_0 \neq 0, \\ &= T \begin{pmatrix} 1 & \log(\lambda/(\lambda - 1)) \\ 0 & 1 \end{pmatrix} T^{-1} & \text{if } \theta_0 = 0, \end{aligned}$$

respectively, where  $\Lambda_0$  and  $T$  are as in Lemma 4.3. From these facts Theorem 2.16 immediately follows.

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DEPARTMENT OF MATHEMATICS, KEIO UNIVERSITY, 3-14-1, HIYOSHI, KOHOKU-KU, YOKOHAMA 223-8522 JAPAN